Measure theory and stochastic processes
TA Session Problems No. 7

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Note: this is only a draft of the solutions discussed on Friday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

Ex. 5.1 (Shreve)

Consider the discounted stock price $D(t)S(t)$ of (5.2.19). In this problem, we derive the formula (5.2.20) for $d\left(D(t)S(t)\right)$ by two methods.

(i) Define $f(x) = S(0)e^x$ and set

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds$$

so that $D(t)S(t) = f\left(X(t)\right)$. Use the Itô-Doeblin formula to compute $df\left(X(t)\right)$.

First, recall the definition of an **Itô process**.

Def. 4.4.3. *Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be an associated filtration. An Itô process is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du, \tag{4.4.16}$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes[1].

Note that (4.4.16) can be also expresses in the differential notation

$$dX(t) = \Delta(t)dW(t) + \Theta(t)dt. \tag{4.4.18}$$

Next, recall the formula for the **quadratic variation of the Itô process**, which describes the rate at which the Itô process accumulates quadratic variation.

Lemma 4.4.4. *The quadratic variation of the Itô process (4.4.16) is*

$$[X, X](t) = \int_0^t \Delta^2(u)du. \tag{4.4.17}$$

We have, therefore that

$$dX(t)dX(t) = \Delta^2(t)dt.$$

Finally, recall the **Itô-Doeblin formula for an Itô process**.

Thm. 4.4.6. *Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous. Then,*

[1] We assume that $\mathbb{E}\int_0^t \Delta^2(u)$ and $\int_0^t |\Theta(u)|du$ are finite for every $t > 0$ so that the integrals on the right-hand side of (4.4.16) are defined and the Itô integral is a martingale.

for every $T \geq 0$,

$$\begin{aligned}
f(T, X(T)) =& f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))dX_t \\
&+ \frac{1}{2}\int_0^T f_{xx}(t, X(t))d[X, X](t) \\
=& f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))\Delta(t)dW(t) \\
&+ \int_0^T f_x(t, X(t))\Theta(t)dt + \frac{1}{2}\int_0^T f_{xx}(t, X(t))\Delta^2(t)dt. \qquad (4.4.22)
\end{aligned}$$

It is easier to remember and use the result of this theorem (4.4.22) if we express it in differential notation as

$$\begin{aligned}
df(t, X(t)) =& f_t(t, X(t))dt + f_x(t, X(t))dX(t) \\
&+ \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t) \qquad (4.4.23) \\
=& f_t(t, X(t))dt + f_x(t, X(t))dX(t) \\
&+ f_x(t, X(t))\Theta(t)dt + \frac{1}{2}f_{xx}(t, X(t))\Delta^2(t)dt. \qquad (4.4.24)
\end{aligned}$$

Recall formula (5.2.16) for the stock price

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}, \qquad (5.2.16)$$

with the differential

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \qquad (5.2.15)$$

and formula (5.2.17) for discount process

$$D(t) = e^{-\int_0^t R(s)ds}, \qquad (5.2.17)$$

with the differential

$$\begin{aligned}
dD(t) &= -R(t)e^{\int_0^t R(s)ds} \\
&= -R(t)D(t)dt. \qquad (5.2.18)
\end{aligned}$$

Then the discounted stock price process is given by formula (5.2.19)

$$D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}, \qquad (5.2.16)$$

and its differential is equal to

$$\begin{aligned}
d(D(t)S(t)) &= (\alpha(t) - R(t))\,D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\
&= \sigma(t)D(t)S(t)\left[\Theta(t)dt + dW(t)\right], \qquad (5.2.20)
\end{aligned}$$

where

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}. \qquad (5.2.21)$$

We need to employ the Itô-Doeblin to show that indeed the differential of (5.2.19) is given by (5.2.20). First, notice that $f(x) = S(0)e^x$, so it does not explicitly depend on time, with $f(x) = f'(x) = f''(x)$ and that $f(X(t)) = S(0)e^{X(t)} = D(t)S(t)$ by (5.2.19). Moreover, we had the following rules

$$\begin{aligned}
dW(t)dW(t) &= dt, \\
dtdW(t) &= 0, \\
dtdt &= 0.
\end{aligned}$$

Notice that $X(t)$ is given by (4.4.16), which is also the exponent in (5.2.16), so that by (4.4.18) we have

$$dX(t) = \sigma(t)dW(t) + \left(\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t)\right) dt, \tag{*}$$

$$dX(t)dX(t) = \sigma(t)dt. \tag{**}$$

Then, by (4.4.22), we can simply write

$$\begin{aligned} d\left(D(t)S(t)\right) &= df(X(t)) \\ &= f'(X(t))dX(t) + \frac{1}{2}f'(X(t))dX(t)dX(t) \\ &= \underbrace{S(0)e^{X(t)}}_{D(t)S(t)} \underbrace{dX(t)}_{(*)} + \frac{1}{2} \underbrace{S(0)e^{X(t)}}_{D(t)S(t)} \underbrace{dX(t)dX(t)}_{(**)} \\ &= D(t)S(t)\left[\sigma(t)dW(t) + \left(\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t)\right) dt + \frac{1}{2}\sigma^2(t)\right] \\ &= D(t)S(t)\left[\sigma(t)dW(t) + (\alpha(t) - R(t)\, dt\right] \\ &= D(t)S(t)\sigma(t)\left[dW(t) + \Theta(t)\right], \end{aligned}$$

which indeed is formula (5.2.20).

(ii) According to Itô's product rule,

$$d\left(D(t)S(t)\right) = S(t)dD(t) + D(t)dS(t) + dD(t)dS(t).$$

Use (5.2.15) and (5.2.18) to work out the right-hand side of this equation.

Recall the **Itô product rule**.

Col. 4.6.3. *Let $X(t)$ and $Y(t)$ be Itô processes. Then*

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t).$$

By (5.2.15) and (5.2.18) we had

$$\begin{aligned} dD(t) &= -R(t)D(t)dt, \\ dS(t) &= \alpha(t)S(t)dt + \sigma(t)S(t)dW(t). \end{aligned}$$

So now we can simply write by the Iô product rule

$$\begin{aligned} d\left(D(t)S(t)\right) =& S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\ =& S(t)\left[-R(t)D(t)dt\right] + D(t)\left[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right] \\ &+ \underbrace{\left[-R(t)D(t)dt\right]\left[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right]}_{=0} \\ =& \left[\alpha(t)D(t)S(t) - R(t)S(t)D(t)\right] dt + \sigma(t)S(t)dW(t) + 0 \\ =& D(t)S(t)\left[\sigma(t)dW(t) + (\alpha(t) - R(t)\, dt\right] \\ =& D(t)S(t)\sigma(t)\left[dW(t) + \Theta(t)\right], \end{aligned}$$

so again we obtained formula (5.2.20).

Ex. 5.8 (Shreve) (Every strictly positive asset is a generalized geometric Brownian motion).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t)$, $0 \leq t \leq T$. Let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by this Brownian motion. Assume there is a unique risk-neutral measure $\tilde{\mathbb{P}}$, and let $\tilde{W}(t)$, $0 \leq t \leq T$, be the Brownian motion under $\tilde{\mathbb{P}}$ obtained by an application of Girsanov's Theorem, Theorem 5.2.3.
Corollary 5.3.2 of the Martingale Representation Theorem asserts that every martingale $\tilde{M}(t)$, $0 \leq t \leq T$, under $\tilde{\mathbb{P}}$ can be written as a stochastic integral with respect to $\tilde{W}(t)$, $0 \leq t \leq T$. In other words, there exists an adapted process $\tilde{\Gamma}$, $0 \leq t \leq T$, such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u) d\tilde{W}(u), \qquad 0 \leq t \leq T$$

Now let $V(T)$ be an almost surely positive ("almost surely" means with probability one under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$ since these two measures are equivalent), $\mathcal{F}(t)$-measurable random variable. According to the risk-neutral pricing formula (5.2.31), the price at time t of a security paying $V(T)$ at time T is

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du} V(T) \middle| \mathcal{F}(t) \right], \qquad 0 \leq t \leq T.$$

First, recall the **Girsanov theorem**.

Thm. 5.2.3. *Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration for this Brownian motion. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process. Define*

$$Z(t) = \exp\left\{ -\int_0^t \Theta(u) dW(u) - \frac{1}{2} \int_0^t \Theta^2(u) du \right\}, \tag{5.2.11}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u) du, \tag{5.2.12}$$

and assume that

$$\mathbb{E} \int_0^T \Theta^2(u) Z^2(u) du < \infty. \tag{5.2.13}$$

Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{F}, \tag{5.2.1}$$

the process $\tilde{W}(t)$, $0 \leq t \leq T$, is a Brownian motion.

Second, recall the **Martingale representation theorem** (MRT).

Thm. 5.3.1. *Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $M(t)$, $0 \leq t \leq T$, be a martingale with respect to this filtration (i.e., for every t, $M(t)$ is $\mathcal{F}(t)$-measurable and for $0 \leq s \leq t \leq T$, $\mathbb{E}[M(t)|\mathcal{F}] = M(s)$). Then there is an adapted process $\Gamma(u)$, $0 \leq u \leq T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u) dW(u), \qquad 0 \leq t \leq T. \tag{5.3.1}$$

Finally, the plan for this exercise can be summarised as follows.

1° When M_t is a positive $\tilde{\mathbb{P}}$-martingale, then we can write

$$dM(t) = M(t) \cdot \frac{1}{M(t)} dM(t). \tag{1}$$

2° Apply MRT to conclude that there exists some adapted process $\tilde{\Gamma}(t)$ such that

$$dM(t) = \tilde{\Gamma}(t) \tilde{W} ds. \tag{2}$$

3° Plug (2) into (1) to obtain

$$dM(t) = M(t)\frac{\tilde{\Gamma}(t)}{M(t)}\tilde{W}ds,$$

as any positive martingale can be expressed as the exponent of an integral w.r.t. the Brownian motion.

4° Add discounting $D(t)$.

5° Apply the Itô product rule.

6° Infer that every positive asset is a generalized (because the volatility may be random) geometric Brownian motion.

(i) Show that there exists an adapted process $\tilde{\Gamma}(t)$, $0 \le t \le T$, such that

$$dV(t) = R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t), \qquad 0 \le t \le T.$$

We had, for $0 \le t \le T$,

$$\begin{aligned} V(t) &= \tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du}V(T)\middle|\,\mathcal{F}(t)\right], \\ D(t) &= e^{\int_0^t -R(u)du}, \end{aligned}$$

so that

$$\begin{aligned} D(t)V(t) &= \tilde{\mathbb{E}}\left[e^{\int_0^t -R(u)du}e^{-\int_t^T R(u)du}V(T)\middle|\,\mathcal{F}(t)\right], \\ &= \tilde{\mathbb{E}}\left[e^{-\int_0^T R(u)du}V(T)\middle|\,\mathcal{F}(t)\right], \\ &= \tilde{\mathbb{E}}\left[D(T)V(T)|\,\mathcal{F}(t)\right], \end{aligned}$$

which means that $D(t)V(t)$ is a $\tilde{\mathbb{P}}$-martingale. Hence, by MRT, there exists an adapted process $\tilde{\Gamma}(t)$, $0 \le t \le T$, such that

$$D(t)V(t) = \int_0^t \tilde{\Gamma}(t)d\tilde{W}(s),$$

which implies

$$\begin{aligned} V(t) &= \frac{1}{D(t)}\int_0^t \tilde{\Gamma}(t)d\tilde{W}(s) \\ &= e^{\int_0^t R(s)ds}\int_0^t \tilde{\Gamma}(t)d\tilde{W}(s). \end{aligned} \tag{3}$$

Next, differentiate both sides of (3) to obtain

$$dV(t) = R(t)D(t)^{-1}\left(\int_0^t \tilde{\Gamma}(s)d\tilde{W}(s)\right)dt + D(t)^{-1}\tilde{\Gamma}(t)d\tilde{W}(t),$$

which means

$$dV(t) = R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t), \tag{4}$$

yielding the required result.

(ii) Show that, for each $t \in [0, T]$, the price of the derivative security $V(t)$ at time t is almost surely positive.
We want to show that for $0 \le t \le T$

$$\tilde{\mathbb{P}}(V(t) > 0) = \tilde{\mathbb{P}}\left(\tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du}V(T)\middle|\,\mathcal{F}(t)\right] > 0\right) = 1.$$

To spare on notation, for the time being fix some t, $0 \le t \le T$, and put

$$\begin{aligned} X &:= X(t) := e^{-\int_t^T R(u)du}V(T), \\ \mathcal{F} &:= \mathcal{F}(t), \\ Y &:= Y(t) := \tilde{\mathbb{E}}[X(t)|\mathcal{F}(t)]. \end{aligned}$$

Clearly,

$$\tilde{\mathbb{P}}(X > 0) = 1,$$

and by the property of conditional expectation

$$\tilde{\mathbb{P}}(\{Y \geq 0\}) = 1.$$

So our goal is to show

$$\tilde{\mathbb{P}}\left(\{Y = 0\}\right) = 0.$$

Denote the above event by A, i.e. $A := \{Y = 0\}$. Naturally, $A \in \mathcal{F}$.
Obviously, $\tilde{\mathbb{E}}[YI_A] = 0$, so we have

$$\begin{aligned} 0 &= \tilde{\mathbb{E}}[YI_A] \\ &\overset{\text{def.Y}}{=} \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[X|\mathcal{F}]I_A] \\ &= \tilde{\mathbb{E}}[XI_A]. \end{aligned}$$

Next, define

$$\begin{aligned} A_0 &:= A \cap \{X \geq 1\}, \\ A_n &:= A \cap \left\{ \frac{1}{n} > X \geq \frac{1}{n+1} \right\}. \end{aligned}$$

We have further that

$$\begin{aligned} 0 &= \tilde{\mathbb{E}}[XI_A] \\ &\overset{\text{lin.}}{=} \tilde{\mathbb{E}}[XI_{A_0}] + \sum_{n=1}^{\infty} \tilde{\mathbb{E}}[XI_{A_n}] \\ &\overset{\text{MI}}{\geq} 1 \cdot \underbrace{\tilde{\mathbb{P}}(A_0)}_{=0} + \sum_{n=1}^{\infty} \frac{1}{n+1} \underbrace{\tilde{\mathbb{P}}(A_n)}_{=0} \\ &= \tilde{\mathbb{P}}(A \cap \{X > 0\}) \\ &= \tilde{\mathbb{P}}(A). \end{aligned}$$

where MI stands for the Markov's inequality[2]. Hence, indeed $\tilde{\mathbb{P}}(A) = 0$, so we have shown that

$$\tilde{\mathbb{P}}(\{Y = 0\}) = 0,$$

which, by equivalence of $\tilde{\mathbb{P}}$ and $\mathbb{P}$, yields that also

$$\mathbb{P}(\{Y = 0\}) = 0.$$

We have put $Y = Y(t) = V(t)$, therefore, indeed, $V(t)$ is a.s. positive.

(iii) Conclude from (i) and (ii) that there exists an adapted process $\sigma(t)$, $0 \leq t \leq T$, such that

$$dV(t) = R(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t), \qquad 0 \leq t \leq T.$$

From the previous point we know that $V > 0$, a.s.. Hence, as we outlined in the beginning

$$\begin{aligned} dV(t) &= V(t)\frac{V(t)}{V(t)}dV(t) \\ &\overset{(4)}{=} V(t)\frac{V(t)}{V(t)}\left(R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t)\right) \\ &= V(t)R(t)dt + V(t)\underbrace{\frac{\tilde{\Gamma}(t)}{V(t)D(t)}}_{:=\sigma(t)}d\tilde{W}(t) \\ &= V(t)R(t)dt + \sigma(t)V(t)d\tilde{W}(t), \end{aligned}$$

which completes the proof and shows that V follows a generalised geometric Brownian motion.

In other words, prior to time T, the price of every asset with almost surely positive price at time T follows a generalized (because the volatility may be random) geometric Brownian motion.

[2]Recall: if X is a nonnegative integrable random variable and $a > 0$, then

$$\mathbb{E}[X] \geq a\mathbb{P}(X \geq a).$$