Measure theory and stochastic processes
TA Session Problems No. 6

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Note: this is only a draft of the solutions discussed on Wednesday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

Ex. 4.9 (Shreve)

For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x, is

$$c(t,x) = xN\left(d_+(T-t,x)\right) - Ke^{-r(T-t)}N\left(d_-(T-t,x)\right),$$

where

$$d_+(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau\right],$$
$$d_-(\tau,x) = d_+(\tau,x) - \sigma\sqrt{\tau},$$

and $N(y)$ is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}dz.$$

The purpose of this exercise is to show that the function c satisfies the Black-Scholes-Merton partial differential equation

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) = rc(t,x), \qquad 0 \le t < T,\ x > 0, \tag{4.10.3}$$

the terminal condition

$$\lim_{t\uparrow T} c(t,x) = (x-K)^+, \qquad x > 0,\ x \neq K, \tag{4.10.4}$$

and the boundary conditions

$$\lim_{t\downarrow 0} c(t,x) = 0, \lim_{x\to\infty}\left[c(t,x) - \left(x - e^{-r(T-t)}K\right)\right] = 0, \qquad 0 \le t < T. \tag{4.10.5}$$

Equation (4.10.4) *and the first part of* (4.10.5) *are usually written more simply but less precisely as*

$$c(T,x) = (x-K)^+, \quad x \ge 0$$

and

$$c(t,0) = 0, \quad 0 \le t < T.$$

For this exercise, we abbreviate $c(t,x)$ as simply c and $d\pm(T-t,x)$ as simply $d_\pm$.

(i) Verify first the equation

$$Ke^{-r(T-t)}N'(d_-) = xN'(d_+) \tag{4.10.6}$$

Since N denotes the standard normal cdf, N' is a pdf of the standard normal distribution, so we have

$$N'(d_-) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_-^2}{2}\right].$$

Next, by definition, d_- is equal to

$$d_-(\tau, x) = d_+(\tau, x) - \sigma\sqrt{\tau},$$

where

$$d_+(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau\right],$$

with the shorthand notation $d_\pm := d_\pm(T-t, x)$. Hence,

$$\begin{aligned}
N'(d_-) &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\left(d_+ - \sigma\sqrt{T-t}\right)^2}{2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_+^2 - 2d_+\sigma\sqrt{T-t} + \sigma^2(T-t)}{2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_+^2}{2}\right] \exp\left[-\frac{-2d_+\sigma\sqrt{T-t}}{2}\right] \exp\left[-\frac{\sigma^2(T-t)}{2}\right] \\
&= N'(d_+) \exp\left[d_+\sigma\sqrt{T-t}\right] \exp\left[-\frac{\sigma^2(T-t)}{2}\right].
\end{aligned}$$

In the middle term we have

$$\begin{aligned}
d_+(T-t, x)\sigma\sqrt{T-t} &= \frac{\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right] \\
&= \log\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t),
\end{aligned}$$

so that this term becomes

$$\begin{aligned}
\exp\left[d_+\sigma\sqrt{T-t}\right] &= \exp\left[\log\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\
&= \frac{x}{K}\exp\left[\left(r + \frac{\sigma^2}{2}\right)(T-t)\right].
\end{aligned}$$

Finally, we arrive at

$$\begin{aligned}
Ke^{-r(T-t)}N'(d_-) &= K\exp\left[-r(T-t)\right]N'(d_+)\frac{x}{K}\exp\left[\left(r + \frac{\sigma^2}{2}\right)(T-t)\right]\exp\left[-\frac{\sigma^2(T-t)}{2}\right] \\
&= K\exp\left[-r(T-t)\right]\frac{x}{K}\exp\left[\left(r + \frac{\sigma^2}{2}\right)(T-t)\right]\exp\left[-\frac{\sigma^2(T-t)}{2}\right]N'(d_+) \\
&= x\exp\left[-r(T-t) + \left(r + \frac{\sigma^2}{2}\right)(T-t) - \frac{\sigma^2(T-t)}{2}\right]N'(d_+) \\
&= x\exp\left[(T-t)\left(-r + \left(r + \frac{\sigma^2}{2}\right) - \frac{\sigma^2}{2}\right)\right]N'(d_+) \\
&= x\exp\left[0\right]N'(d_+) \\
&= xN'(d_+),
\end{aligned}$$

which completes the proof

(ii) *Show that* $c_x = N(d_+)\cdot$ *This is the* delta *of the option. (Be careful! Remember that* d_+ *is a function of* x.*)*

The Black-Scholes-Merton price at time t is given by

$$c := c(t, x) = xN\left(d_+(T-t, x)\right) - Ke^{-r(T-t)}N\left(d_-(T-t, x)\right),$$

so

$$c_x = N\left(d_+(T-t,x)\right) + x\frac{\partial}{\partial x}\Big[N\left(d_+(T-t,x)\right)\Big] - xKe^{-r(T-t)}\frac{\partial}{\partial x}\Big[N\left(d_-(T-t,x)\right)\Big].$$

For the terms $\frac{\partial}{\partial x}\Big[N\left(d_\pm(T-t,x)\right)\Big]$ we have

$$\begin{aligned}\frac{\partial}{\partial x}\Big[N\left(d_\pm(T-t,x)\right)\Big] &= N'(d_\pm(T-t,x))\frac{\partial}{\partial x}d_\pm(T-t,x)\\ &= N'(d_\pm(T-t,x))\frac{\partial}{\partial x}d_+(T-t,x)\\ &= N'(d_\pm(T-t,x))\frac{1}{\sigma\sqrt{T-t}}\frac{1}{K}\frac{K}{x}\\ &= N'(d_\pm(T-t,x))\frac{1}{\sigma\sqrt{T-t}}\frac{1}{x},\end{aligned}$$

so we can write

$$\begin{aligned}c_x &= N\left(d_+(T-t,x)\right) + xN'(d_+(T-t,x))\frac{1}{\sigma\sqrt{T-t}}\frac{1}{x} - \underbrace{xKe^{-r(T-t)}N'(d_-(T-t,x))}_{(i)}\frac{1}{\sigma\sqrt{T-t}}\frac{1}{x}\\ &= N\left(d_+(T-t,x)\right) + xN'(d_+(T-t,x))\frac{1}{\sigma\sqrt{T-t}}\frac{1}{x} - xN'\left(d_+(T-t,x)\right)\frac{1}{\sigma\sqrt{T-t}}\frac{1}{x}\\ &= N\left(d_+(T-t,x)\right),\end{aligned}$$

which is the desired result.

(iii) Show that

$$c_t = -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+).$$

This is the theta *of the option.*

The Black-Scholes-Merton price at time t is given by

$$c := c(t,x) = xN\left(d_+(T-t,x)\right) - Ke^{-r(T-t)}N\left(d_-(T-t,x)\right),$$

so

$$\begin{aligned}c_t =& xN'\left(d_+(T-t,x)\right)\frac{\partial}{\partial t}d_+(T-t,x)\\ &- rKe^{-r(T-t)}N\left(d_-(T-t,x)\right) - Ke^{-r(T-t)}N'\left(d_-(T-t,x)\right)\frac{\partial}{\partial t}d_-(T-t,x).\end{aligned}$$

For the term $\frac{\partial}{\partial t}d_-(T-t,x)$ we have

$$\frac{\partial}{\partial t}d_-(T-t,x) = \frac{\partial}{\partial t}d_+(T-t,x) + \frac{\sigma}{2\sqrt{T-t}},$$

so we can write

$$\begin{aligned}c_t =& xN'\left(d_+(T-t,x)\right)\frac{\partial}{\partial t}d_+(T-t,x)\\ &- rKe^{-r(T-t)}N\left(d_-(T-t,x)\right) - \underbrace{Ke^{-r(T-t)}N'\left(d_-(T-t,x)\right)}_{(i)}\frac{\partial}{\partial t}d_-(T-t,x)\\ =& xN'\left(d_+(T-t,x)\right)\frac{\partial}{\partial t}d_+(T-t,x)\\ &- rKe^{-r(T-t)}N\left(d_-(T-t,x)\right) - xN'(d_+)\left[\frac{\partial}{\partial t}d_+(T-t,x) + \frac{\sigma}{2\sqrt{T-t}}\right]\\ =& - rKe^{-r(T-t)}N\left(d_-(T-t,x)\right) - xN'(d_+)\frac{\sigma}{2\sqrt{T-t}},\end{aligned}$$

which is the required formula.

(iv) *Use the formulas above to show that c satisfies* (4.10.3).

We had that

$$c_x = N\left(d_+(T-t,x)\right),$$

so

$$c_{xx} = N'\left(d_+(T-t,x)\right)\frac{\partial}{\partial x}d_+(T-t,x).$$

Plugging of all the above results in (4.10.3) gives

$$c_t + rxc_x + \frac{1}{2}\sigma^2x^2c_{xx} = -rKe^{-r(T-t)}N\left(d_-(T-t,x)\right) \underline{-xN'(d_+(T-t,x))\frac{\sigma}{2\sqrt{T-t}}} + rxN\left(d_+(T-t,x)\right)$$
$$\underline{+\frac{1}{2}\sigma^2x^2N'\left(d_+(T-t,x)\right)\frac{\partial}{\partial x}d_+(T-t,x).}$$

Now consider the underlined terms. We have

$$\begin{aligned}
-xN'(d_+)\frac{\sigma}{2\sqrt{T-t}} + \frac{1}{2}\sigma^2x^2N'\left(d_+\right)\frac{\partial}{\partial x}d_+ &= \frac{\sigma}{2}xN'(d_+)\left(\sigma x\frac{\partial d_+}{\partial x} - \frac{1}{\sqrt{T-t}}\right)\\
&= \frac{\sigma}{2}xN'(d_+)\left(\sigma x\frac{1}{\sigma\sqrt{T-t}x} - \frac{1}{\sqrt{T-t}}\right)\\
&= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
c_t + rxc_x + \frac{1}{2}\sigma^2x^2c_{xx} &= rxN\left(d_+(T-t,x)\right) - rKe^{-r(T-t)}N\left(d_-(T-t,x)\right)\\
&= r\left[xN\left(d_+(T-t,x)\right) - Ke^{-r(T-t)}N\left(d_-(T-t,x)\right)\right]\\
&= rc(t,x),
\end{aligned}$$

which completes the proof.

(v) *Show that for $x > K$, $\lim_{t\uparrow T} d_\pm = \infty$, but for $0 < x < K$, $\lim_{t\uparrow T} d_\pm = -\infty$. Use these equalities to derive the terminal condition* (4.10.4).

We need to arrive at

$$\lim_{t\uparrow T} c(t,x) = (x-K)^+, \qquad x > 0,\ x \neq K.$$

Recall,

$$\begin{aligned}
d_+(T-t,x) &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right],\\
d_-(T-t,x) &= d_+(T-t,x) - \sigma\sqrt{T-t}\\
&= \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right],\\
d_\pm(T-t,x) &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right].
\end{aligned}$$

Notice that if $x > K$, then $\frac{x}{K} > 1$, so $\log\frac{x}{K} > 0$ and $d_+ > 0$; if $x < K$, then $\log\frac{x}{K} < 0$. When $t \uparrow T$ then $\sqrt{T-t} \downarrow 0$, so we have

$$\begin{aligned}
\lim_{t\uparrow T} d_\pm(T-t,x) &= \lim_{t\uparrow T}\frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right]\\
&= \lim_{t\uparrow T}\frac{1}{\sigma\sqrt{T-t}}\log\frac{x}{K} + \lim_{t\uparrow T}\frac{1}{\sigma\sqrt{T-t}}\left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\\
&= \lim_{t\uparrow T}\frac{1}{\sigma\sqrt{T-t}}\log\frac{x}{K} + \underbrace{\lim_{t\uparrow T}\frac{\sqrt{T-t}}{\sigma}\left(r \pm \frac{1}{2}\sigma^2\right)}_{=0}\\
&= \begin{cases}\infty, & \text{if } x > K,\\ -\infty, & \text{if } x < K.\end{cases}
\end{aligned}$$

Next,

$$\lim_{t \uparrow T} N\left(d_{\pm}(T-t,x)\right) = \begin{cases} 1, & \text{if } x > K, \\ 0, & \text{if } x < K. \end{cases}$$

Finally,

$$\begin{aligned}
\lim_{t \uparrow T} c\left(t,x\right) &= \lim_{t \uparrow T} \Big(xN\left(d_{+}(T-t,x)\right) - Ke^{-r(T-t)} N\left(d_{-}(T-t,x)\right) \Big) \\
&= \begin{cases} x \cdot 1 - Ke^{-r(T-T^{-})} \cdot 1, & \text{if } x > K, \\ x \cdot 0 - Ke^{-r(T-T^{-})} \cdot 0, & \text{if } x < K, \end{cases} \\
&= \begin{cases} x - K, & \text{if } x > K, \\ 0, & \text{if } x < K, \end{cases} \\
&= (x-K)^{+},
\end{aligned}$$

which establishes the terminal condition (4.10.4).

(vi) Show that for $0 \le t < T$, $\lim_{x \downarrow 0} d_{\pm} = -\infty$. Use this fact to verify the first part of boundary condition (4.10.5) as $x \downarrow 0$.

Let $0 \le t < T$. When $x \downarrow 0$ then $\log \frac{x}{K} \to -\infty$ and

$$\begin{aligned}
\lim_{x \downarrow 0} d_{\pm}(T-t,x) &= \lim_{x \downarrow 0} \frac{1}{\sigma\sqrt{T-t}} \left[\log \frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^{2} \right)(T-t) \right] \\
&= -\infty,
\end{aligned}$$

so that

$$\lim_{x \downarrow 0} N\left(d_{\pm}(T-t,x)\right) = 0.$$

Thus,

$$\begin{aligned}
\lim_{x \downarrow 0} c\left(t,x\right) &= \lim_{x \downarrow 0} \Big(xN\left(d_{+}(T-t,x)\right) - Ke^{-r(T-t)} N\left(d_{-}(T-t,x)\right) \Big) \\
&= 0,
\end{aligned}$$

establishing the first part of the boundary condition (4.10.5).

(vii) Show that for $0 \le t < T$, $\lim_{x \to \infty} d_{\pm} = \infty$. Use this fact to verify the second part of boundary condition (4.10.5) as $x \to \infty$. In this verification, you will need to show that

$$\lim_{x \to \infty} \frac{N(d_{+}) - 1}{x^{-1}} = 0.$$

This is an indeterminate form $\frac{0}{0}$, and L'Hôpital's rule implies that this limit is

$$\lim_{x \to \infty} \frac{\frac{d}{dx}\left[N(d_{+}) - 1\right]}{\frac{d}{dx} x^{-1}}.$$

Work out this expression and use the fact that

$$x = K \exp\left\{ \sigma\sqrt{T-t}\, d_{+} - (T-t)\left(r + \frac{1}{2}\sigma^{2} \right) \right\}$$

to write this expression solely in terms of d_{+} (i.e., without the appearance of any x except the x in the argument of $d_{+}(T-t,x)$). Then argue that the limit is zero as $d_{+} \to \infty$.

Let $0 \le t < T$. We need to show that

$$\lim_{x \to \infty} \left[c(t,x) - \left(x - e^{-r(T-t)} K \right) \right] = 0.$$

When $x \to \infty$ then $\log \frac{x}{K} \to \infty$ and

$$\begin{aligned}
\lim_{x \to \infty} d_{\pm}(T-t,x) &= \lim_{x \to \infty} \frac{1}{\sigma\sqrt{T-t}} \left[\log \frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^{2} \right)(T-t) \right] \\
&= \infty,
\end{aligned}$$

so that

$$\lim_{x\to\infty} N\left(d_{\pm}(T-t,x)\right) = 1.$$

Next,

$$\begin{aligned}
\lim_{x\to\infty}\left[c(t,x)-\left(x-e^{-r(T-t)}K\right)\right] &= \lim_{x\to\infty}\left[xN\left(d_+(T-t,x)\right)-Ke^{-r(T-t)}N\left(d_-(T-t,x)\right)-\left(x-e^{-r(T-t)}K\right)\right]\\
&= \lim_{x\to\infty}\left[xN\left(d_+(T-t,x)\right)-x\right]\\
&= \lim_{x\to\infty} x\left[N\left(d_+(T-t,x)\right)-1\right]\\
&= \lim_{x\to\infty}\frac{N\left(d_+(T-t,x)\right)-1}{x^{-1}}. \qquad (1)
\end{aligned}$$

If the above limit exists, it is equal to

$$\begin{aligned}
\lim_{x\to\infty}\frac{N\left(d_+(T-t,x)\right)-1}{x^{-1}} &= \lim_{x\to\infty}\frac{\frac{d}{dx}\left[N\left(d_+(T-t,x)\right)-1\right]}{-x^{-2}}\\
&= \lim_{x\to\infty}\frac{N'\left(d_+(T-t,x)\right)\frac{\partial}{\partial x}d_+(T-t,x)}{-x^{-2}}\\
&= \lim_{x\to\infty}\frac{N'\left(d_+(T-t,x)\right)\frac{1}{x\sigma\sqrt{T-t}}}{-x^{-2}}\\
&= \lim_{x\to\infty}-\frac{N'\left(d_+(T-t,x)\right)\frac{1}{\sigma\sqrt{T-t}}}{x^{-1}}\\
&= -\frac{1}{\sigma\sqrt{T-t}}\lim_{x\to\infty} xN'\left(d_+(T-t,x)\right)\\
&= -\frac{1}{\sigma\sqrt{T-t}}\lim_{x\to\infty} x\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{d_+^2(T-t,x)}{2}\right\}\\
&= -\frac{1}{\sigma\sqrt{2\pi(T-t)}}\lim_{x\to\infty} x\exp\left[-\frac{d_+^2(T-t,x)}{2}\right]\\
&\overset{(*)}{=} -\frac{1}{\sigma\sqrt{2\pi(T-t)}}\lim_{d_+\to\infty} K\exp\left[\sigma\sqrt{T-t}d_+-\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]\exp\left[-\frac{d_+^2}{2}\right]\\
&= -\frac{K}{\sigma\sqrt{2\pi(T-t)}}\exp\left[-\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]\lim_{d_+\to\infty}\exp\left[\sigma\sqrt{T-t}d_+-\frac{d_+^2}{2}\right]\\
&= 0,
\end{aligned}$$

where in $(*)$ we used the hint and expressed x using the formula for d_+ as follows

$$\begin{aligned}
d_+ &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K}+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right],\\
x &= K\exp\left[\sigma\sqrt{T-t}d_+-\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]
\end{aligned}$$

as well as took the limit w.r.t. d_+, since when $x\to\infty$ also $d_+\to\infty$. So the limit in (1) exists and is equal to 0, which finally establishes the second boundary condition in (4.10.5).

Ex. 4.14 (Shreve)

In the derivation of the Itô-Doeblin formula, Theorem 4.4.1, we considered only the case of the function $f(x) = \frac{1}{2}x^2$*, for which* $f"(x) = 1$*. This made it easy to determine the limit of the last term,*

$$\frac{1}{2}\sum_{j=0}^{n-1} f''(W(t_j))\left[W(t_{j+1}) - W(t_j)\right]^2,$$

appearing in (4.4.5). Indeed,

$$\begin{aligned}\lim_{||\Pi\to 0||}\sum_{j=0}^{n-1} f''(W(t_j))\left[W(t_{j+1}) - W(t_j)\right]^2 &= \lim_{||\Pi\to 0||}\sum_{j=0}^{n-1}\left[W(t_{j+1}) - W(t_j)\right]^2\\ &= [W,W](T) = T\\ &= \int_0^T f''(W(t))dt.\end{aligned}$$

If we had been working with an arbitrary function $f(x)$*, we could not replace* $f"(W(t_i))$ *by* 1 *in the argument above. It is tempting in this case to just argue that* $[W(t_{j+1}) - W(t_j)]^2$ *is approximately equal to* $t_{j+1} - t_j$*, so that*

$$\sum_{j=0}^{n-1} f''(W(t_j))\left[W(t_{j+1}) - W(t_j)\right]^2$$

is approximately equal to

$$\sum_{j=0}^{n-1} f''(W(t_j))(t_{j+1} - t_j)$$

and this has limit $\int_0^T f''(W(t))dt$ *as* $||\Pi|| \to 0$*. However, as discussed in Remark 3.4.4, it does not make sense to say that* $[W(t_{j+1}) - W(t_j)]^2$ *is approximately equal to* $t_{j+1} - t_j$*. In this exercise, we develop a correct explanation for the equation*

$$\lim_{||\Pi\to 0||}\sum_{j=0}^{n-1} f''(W(t_j))\left[W(t_{j+1}) - W(t_j)\right]^2 = \int_0^T f''(W(t))dt. \tag{4.10.22}$$

Define

$$Z_j = f''(W(t_j))\left[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\right]$$

so that

$$\sum_{j=0}^{n-1} f''(W(t_j))\left[W(t_{j+1}) - W(t_j)\right]^2 = \sum_{j=0}^{n-1} Z_j + \sum_{j=0}^{n-1} f''(W(t_j))(t_{j+1} - t_j)^2. \tag{4.10.23}$$

For completeness, recall the mentioned remark.

Rem. 3.4.4. *In the proof above, we derived the equations (3.4.6) and (3.4.7):*

$$\mathbb{E}\left[(W(t_{j+1} - W(t_j)))^2\right] = t_{j+1} - t_j$$

and

$$Var\left[(W(t_{j+1} - W(t_j)))^2\right] = 2\left(t_{j+1} - t_j\right)^2.$$

It is tempting to argue that when $t_{j+1} - t_j$ *is small,* $(t_{j+1} - t_j)^2$ *is very small, and therefore* $(W(t_{j+1}) - W(t_j))^2$*, although random, is with high probability near its mean* $t_{j+1} - t_j$*. We could therefore claim that*

$$(W(t_{j+1}) - W(t_j))^2 \approx t_{j+1} - t_j \tag{3.4.8}$$

This approximation is trivially true because, when $t_{j+1} - t_j$ *is small, both sides are near zero. It would also be true if we squared the right-hand side, multiplied the right-hand side by 2, or made any of several other* *significant changes to the right-hand side. In other words, (3.4.8) really has no content. A better way to try to capture what we think is going on is to write*

$$(W(t_{j+1}) - W(t_j))^2 \approx t_{j+1} - t_j \tag{3.4.9}$$

instead of (3.4.8). However,

$$\frac{(W(t_{j+1}) - W(t_j)))^2}{t_{j+1} - t_j}$$

is in fact not near 1, regardless of how small we make $t_{j+1} - t_j$. ti. It is the square of the standard normal random variable

$$Y_{j+1} = \frac{W(t_{j+1}) - W(t_j)}{\sqrt{t_{j+1} - t_j}}$$

and its distribution is the same, no matter how small we make $t_{j+1} - t_j$.
(...)
We write informally

$$dW(t)dW(t) = dt \tag{3.4.10}$$

but this should not be interpreted to mean either (3.4.8) or (3.4.9).
(...)
we conclude that

Brownian motion accumulates quadratic variation at rate one per unit time.

(i) Show that Z_j is $\mathcal{F}(t_{j+1})$-measurable and

$$\mathbb{E}\left[Z_j|\mathcal{F}(t_j)\right] = 0, \quad \mathbb{E}\left[Z_j^2|\mathcal{F}(t_j)\right] = 2\left[f''(W(t_j))\right]^2 (t_{j+1} - t_j)^2.$$

First, notice that $Z_j \in \mathcal{F}(t_{j+1})$ and that the Brownian motion increment $W(t_{j+1}) - W(t_j) \perp \mathcal{F}(t_j)$, with the vaiance $t_{j+1} - t_j$. Then, we have

$$\begin{aligned}
\mathbb{E}\left[Z_j|\mathcal{F}(t_j)\right] &= f''(W(t_j))\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\middle|\mathcal{F}(t_j)\right] \\
&= f''(W(t_j))\left[(t_{j+1} - t_j) - (t_{j+1} - t_j)\right] \\
&= 0.
\end{aligned}$$

Next, recall that for $X \sim N(0, s)$ we have $\mathbb{E}X^4 = 3\mathbb{E}[X^2]^2 = 3s^2$. Then.

$$\begin{aligned}
\mathrm{Var}\left[Z_j|\mathcal{F}(t_j)\right] =& \left[f''(W(t_j))\right]^2 \mathbb{E}\left[\left[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\right]^2\middle|\mathcal{F}(t_j)\right] \\
=& \left[f''(W(t_j))\right]^2 \mathbb{E}\left[(W(t_{j+1}) - W(t_j))^4 - 2\left(W(t_{j+1}) - W(t_j)\right)^2 (t_{j+1} - t_j) + (t_{j+1} - t_j)^2\middle|\mathcal{F}(t_j)\right] \\
\overset{\text{lin.}}{=}& \left[f''(W(t_j))\right]^2 \left[\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^4\middle|\mathcal{F}(t_j)\right] - 2\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^2 (t_{j+1} - t_j)\middle|\mathcal{F}(t_j)\right]\right. \\
&\left. + \mathbb{E}\left[(t_{j+1} - t_j)^2\middle|\mathcal{F}(t_j)\right]\right] \\
=& \left[f''(W(t_j))\right]^2 \left[3(t_{j+1} - t_t)^2 - 2(t_{j+1} - t_t)^2 + (t_{j+1} - t_t)^2\right] \\
=& 2\left[f''(W(t_j))\right]^2 (t_{j+1} - t_t)^2,
\end{aligned}$$

which completes the proof.

It remains to show that

$$\lim_{||\Pi \to 0||} \sum_{j=0}^{n-1} Z_j = 0 \tag{4.10.24}$$

This will cause us to obtain (4.10.22) when we take the limit in (4.10.23). Prove (4.10.24) in the following steps.

(iii) Show that $\mathbb{E}\sum_{j=0}^{n-1} Z_j = 0$.

Using interated conditioning and linearity of conditional expectation we can write

$$\begin{aligned}\mathbb{E}\sum_{j=0}^{n-1} Z_j &= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{E}\left[Z_j|\mathcal{F}(t_j)\right]\right]\\ &\overset{(i)}{=} \mathbb{E}\left[\sum_{j=0}^{n-1} 0\right]\\ &= 0,\end{aligned}$$

which is the required result.

(iv) Under the assumption that $\mathbb{E}\int_0^T \left[f''(W(t))\right]^2 dt$ *is finite, show that*

$$\lim_{||\Pi||\to 0} Var\left[\sum_{j=0}^{n-1} Z_j\right] = 0.$$

(Warning: The random variables $Z_1, Z_2, \ldots, Z_{n-1}$ *opare not independent.)*

$$\begin{aligned}\text{Var}\left[\sum_{j=0}^{n-1} Z_j\right] &= \mathbb{E}\left[\left(\sum_{j=0}^{n-1} Z_j\right)^2\right]\\ &= \mathbb{E}\left[\sum_{j=0}^{n-1} Z_j^2 + 2\sum_{0\le i<j\le n} Z_i Z_j\right]\\ &\overset{\text{lin.}}{\underset{\text{IC}}{=}} \sum_{j=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[Z_j^2\middle|\mathcal{F}(t_j)\right]\right] + 2\sum_{0\le i<j\le n}\mathbb{E}\left[Z_i\mathbb{E}\left[Z_j^2\middle|\mathcal{F}(t_j)\right]\right]\\ &= \sum_{j=0}^{n-1}\mathbb{E}\left[2\left[f''(W(t_j))\right]^2(t_{j+1}-t_j)^2\right]\\ &= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2\mathbb{E}\left[\left[f''(W(t_j))\right]^2\right]\\ &\le 2\underbrace{\max_{0\le j\le n-1}|t_{j+1}-t_j|}_{\to 0}\underbrace{\sum_{j=0}^{n-1}\mathbb{E}\left[\left[f''(W(t_j))\right]^2\right](t_{j+1}-t_j)}_{(**)}\\ &\to 0,\end{aligned}$$

where in $(**)$ we use the assumption that $\mathbb{E}\int_0^T \left[f''(W(t))\right]^2 dt < \infty$ as then

$$\begin{aligned}\lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}\mathbb{E}\left[\left[f''(W(t_j))\right]^2\right](t_{j+1}-t_j) &= \lim_{||\Pi||\to 0}\mathbb{E}\left[\sum_{j=0}^{n-1}\left[f''(W(t_j))\right]^2(t_{j+1}-t_j)\right]\\ &= \mathbb{E}\int_0^T \left[f''(W(t))\right]^2 dt\\ &< \infty.\end{aligned}$$

This completes the proof.

From *(iii)*, we conclude that $\sum_{j=0}^{n-1} Z_j = 0$ converges to its mean, which by *(ii)* is zero. This establishes (4.10.24).