Note: this is only a draft of the solutions discussed on Wednesday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

**Ex. 4.1 (Shreve)**

Suppose $M(t)$, $0 \leq t \leq T$ is a martingale with respect to some filtration $\mathcal{F}(t)$, $0 \leq t \leq T$. Let $\Delta(t)$, $0 \leq t \leq T$, be a simple process adapted to $\mathcal{F}(t)$ (i.e., there is a partition $\Pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$ such that, for every $j$, $\Delta(t_j)$ is $\mathcal{F}(t_j)$-measurable and $\Delta(t)$ is constant in $t$ on each subinterval $[t_j, t_{j+1})$). For $t \in [t_k, t_{k+1})$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j) [M(t_{j+1}) - M(t_j)] + \Delta(t_k) [M(t) - M(t_k)].$$

(1)

We think of $M(t)$ as the price of an asset at time $t$ and $\Delta(t_j)$ as the number of shares of the asset held by an investor between times $t_j$ and $t_{j+1}$. Then $I(t)$ is the capital gains that accrue to the investor between times 0 and $t$. Show that $I(t)$, $0 \leq t \leq T$, is a martingale.

First, recall the definition of a **martingale**.

**Def 2.3.5(i).** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $T$ be a fixed positive number, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be a filtration of sub-$\sigma$-algebras of $\mathcal{F}$. Consider an adapted stochastic process $M(t)$, $0 \leq t \leq T$. If

$$E[M(t) | \mathcal{F}(s)] = M(s), \quad \text{for all } 0 \leq s \leq t \leq T,$$

we say this process is a martingale. It has no tendency to rise or fall.

Next, recall two important notions. Below we let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$, where $T > 0$ is fixed, i.e. $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$.

A **simple process** $\Delta(t)$ is an adapted stochastic process, which is constant in $t$ on each subinterval $[t_j, t_{j+1})$.

The **Itô integral of a simple process** $\Delta(t)$ is a stochastic process given by

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j) [W(t_{j+1}) - W(t_j)] + \Delta(t_k) [W(t) - W(t_k)],$$

(4.2.2)

where $t_k \leq t \leq t_{k+1}$, which is denoted as

$$I(t) = \int_0^t \Delta(u) dW(u).$$

Finally, recall that the Itô integral is a **martingale**.

**Thm. 4.3.1.** The Itô integral defined by (4.2.2) is a martingale.
Let \( 0 \leq s \leq t \leq T \) and wlog\(^1\) assume \( s = t_l \) and \( t = t_k \), for some \( l, k \). We need to check what the expectations of \( I(t) \) (1) given \( \mathcal{F}(s) \) is. We have

\[
\mathbb{E}[I(t)|\mathcal{F}(s)] = \mathbb{E}[I(t_k)|\mathcal{F}(t_l)] \\
= \mathbb{E}\left[ \sum_{j=0}^{k-1} \Delta(t_j) [M(t_{j+1}) - M(t_j)] \bigg| \mathcal{F}(t_l) \right] \\
= \mathbb{E}\left[ \sum_{j=0}^{l-1} \Delta(t_j) [M(t_{j+1}) - M(t_j)] + \sum_{j=l}^{k-1} \Delta(t_j) [M(t_{j+1}) - M(t_j)] \bigg| \mathcal{F}(t_l) \right] \\
= \lim_{l \to 0} \left[ \sum_{j=0}^{l-1} \mathbb{E}[\Delta(t_j) [M(t_{j+1}) - M(t_j)] | \mathcal{F}(t_l)] + \sum_{j=l}^{k-1} \mathbb{E}[\Delta(t_j) [M(t_{j+1}) - M(t_j)] | \mathcal{F}(t_l)] \right] \\
= I(s) + \sum_{j=0}^{k-1} \mathbb{E}[\Delta(t_j) [M(t_{j+1}) - M(t_j)] | \mathcal{F}(t_l)] \\
= I(s) + \mathbb{E}[0|\mathcal{F}(t_l)] \\
= I(s),
\]

where IC denotes iterated conditioning\(^2\), which shows that \( I(t) \) is a martingale.

Notice that the “trick” with iterated conditioning allowed us to make use of the martingale property of the process \( M \), i.e. we could write that

\[
\mathbb{E}[\Delta(t_j) [M(t_{j+1}) - M(t_j)] | \mathcal{F}(t_j)] = \Delta(t_j) \mathbb{E}[M(t_{j+1}) | \mathcal{F}(t_j)] - \Delta(t_j) M(t_j) \\
= \Delta(t_j) M(t_j) - \Delta(t_j) M(t_j) \\
= 0.
\]

---

\(^1\)Without loss of generality. Indeed, as we can always take a new partition of \([0, T]\) with re-arranged indices.

\(^2\) Cf. Thm. 2.3.2(iii): If \( \mathcal{H} \) is a sub-\(\sigma\)-algebra of \( \mathcal{G} \) (\( \mathcal{H} \) contains less information than \( \mathcal{G} \)) and \( X \) is an integrable random variable, then

\[
\mathbb{E}[\mathbb{E}[X|\mathcal{H}] | \mathcal{H}] = \mathbb{E}[X|\mathcal{H}].
\] (2.3.20)
Before moving to the next exercise, let us go through a short recap on convergence and general Itô integrals.

First, recall the definition of convergence in the $p$-th moment.

**Def.** Let $(X_n)_{n=1}^\infty$ be a sequence of random variables and $X$ be a random variable defined on the same probability space. We say that $(X_n)_{n=1}^\infty$ converges to $X$ in the $p$-th moment (in $L^p$), $0 < p < \infty$, if $E|X|^p < \infty$, $E|X_n|^p < \infty$, $\forall n$, and

$$\lim_{n \to \infty} E|X_n - X|^p = 0,$$

and we denote this by $X_n \overset{L^p}{\to} X$.

Second, recall the **Itô isometry** property of the Itô integral (4.2.2).

**Thm. 4.2.2.** The Itô integral defined by (4.2.2) satisfies

$$E I^2(t) = E \int_0^t \Delta^2(u) du. \quad (4.2.6.)$$

Formula (4.2.6.) allows us to compute $\text{Var}I(t) = E I^2(t)$, where the latter equality follows from the fact that $E I(t) = 0$, $\forall t \geq 0$.

Next, for a general integrand, being an adapted stochastic process $\Delta(t)$, its Itô integral is constructed by approximating $\Delta(t)$ by simple processes $\Delta_n(t)$. The latter are chosen in such a way that they converge to the continuously varying $\Delta(t)$, which means that

$$\lim_{t \to \infty} E \int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0. \quad (4.3.2)$$

More formally, the **Itô integral for the continuously varying integrand** $\Delta(t)$ is defined by the formula

$$I(t) = \int_0^t \Delta(u) dW(u) := \lim_{n \to \infty} \int_0^t \Delta_n(u) dW(u), \quad 0 \leq t \leq T. \quad (4.3.3)$$

For each $t$, the limit in (4.3.3) exists because $I_n(t) = \int_0^t \Delta_n(u) dW(u)$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})^3$.

This is because of Itô’s isometry (Thm. 4.2.2), which yields

$$E (I_n(t) - I_m(t))^2 = E \int_0^t |\Delta_n(u) - \Delta_m(u)|^2 du.$$

As a consequence of (4.3.2), the right-hand side has limit zero as $n$ and $m$ approach infinity.

Finally, recall the **properties of the Itô integral**.

**Thm. 4.3.1.** Let $T$ be a positive constant and let $\Delta(t)$, $0 \leq t \leq T$, be an adapted stochastic process that satisfies

$$E \int_0^T \Delta^2(t) dt < \infty. \quad (4.3.1)$$

Then $I(t) = \int_0^t \Delta(u) dW(u)$ defined by (4.3.3) has the following properties.

- (a) **(Continuity)** As a function of the upper limit of integration $t$, the paths of $I(t)$ are continuous.
- (b) **(Adaptivity)** For each $t$, $I(t)$ is $\mathcal{F}(t)$-measurable.
- (c) **(Linearity)** If $I(t) = \int_0^t \Delta(u) dW(u)$ and $J(t) = \int_0^t \Gamma(u) dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u)) dW(u)$; furthermore, for every constant $c$, $cI(t) = \int_0^t c\Delta(u) dW(u)$.
- (d) **(Martingale)** $I(t)$ is a martingale.
- (e) **(Itô isometry)** $E I^2(t) = E \int_0^t \Delta^2(u) du$.
- (f) **(Quadratic variation)** $[I, I](t) = \int_0^t \Delta^2(u) du$.

---

For $0 \leq \infty$, the $L^p$ spaces are complete (when equipped with an appropriate norm).
Let $W(t)$, $t \geq 0$, be a Brownian motion. Let $T$ be a fixed positive number and let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = T$). For each $j$, define $t'_j = \frac{t_j + t_{j+1}}{2}$ to be the midpoint of the interval $[t_j, t_{j+1}]$.

(i) Define the half-sample quadratic variation corresponding to $\Pi$ to be

$$Q_{\Pi}/2 = \sum_{j=0}^{n-1} (W(t'_j) - W(t_j))^2.$$  

Show that $Q_{\Pi}/2$ has limit $\frac{1}{2}T$ as $||\Pi|| \to 0$.

(Hint: It suffices to show that $\mathbb{E}Q_{\Pi}/2 = \frac{1}{2}T$ and $\lim_{||\Pi|| \to 0} \text{Var}(Q_{\Pi}/2) = 0$.)

In this exercise we will consider convergence in $L^2$, since the Stratonovich integral is defined as the limit in $L^2$ (similarly to the Itô integral).

Using the hint we can start with computing the expected value of the half-sample quadratic variation under consideration. We have

$$\mathbb{E}(Q_{\Pi}/2) = \mathbb{E}\left[\sum_{j=0}^{n-1} (W(t'_j) - W(t_j))^2\right]$$

$$= \sum_{j=0}^{n-1} \mathbb{E}\left[(W(t'_j) - W(t_j))^2\right]$$

$$= \sum_{j=0}^{n-1} (t'_j - t_j)$$

$$= \sum_{j=0}^{n-1} \left(\frac{t_j + t_{j+1}}{2} - t_j\right)$$

$$= \sum_{j=0}^{n-1} \frac{t_{j+1} - t_j}{2}$$

$$= \frac{T}{2},$$

where in $(*)$ we use the fact that for $0 \leq s \leq t$ the increment of the Brownian motion $W(t) - W(s) \sim N(0, t - s)$. Notice that in the last step we used the following equality (we will use it in the next point)

$$\frac{T}{2} = \sum_{j=0}^{n-1} \frac{t_{j+1} - t_j}{2}.$$
Next, for the variance we have
\[
\text{Var}(Q_{1/2}) = \mathbb{E}\left[ \left( \sum_{j=0}^{n-1} (W(t_j^*) - W(t_j))^2 - \frac{T}{2} \right)^2 \right] \\
= \mathbb{E}\left[ \left( \sum_{j=0}^{n-1} (W(t_j^*) - W(t_j))^2 - \sum_{j=0}^{n-1} \frac{t_{j+1} - t_j}{2} \right)^2 \right] \\
= \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \mathbb{E}\left[ \left( (W(t_j^*) - W(t_j))^2 - \frac{t_{j+1} - t_j}{2} \right) \left( (W(t_k^*) - W(t_k))^2 - \frac{t_{k+1} - t_k}{2} \right) \right] \\
= \sum_{j=0}^{n-1} \mathbb{E}\left[ (W(\tilde{t}_j))^2 - \frac{t_{j+1} - t_j}{2} \right]^2 \\
\leq \frac{T}{2} \max_{1 \leq j \leq n} |t_{j+1} - t_j| \\
\to 0,
\]
where \( \tilde{t}_j = \frac{t_{j+1} - t_j}{2} \) and (**) is because of
\[
\mathbb{E}\left[ (W^2(t) - t)^2 \right] = \mathbb{E}\left[ W^4(t) - 2tW^2(t) + t^2 \right] \\
= \mathbb{E}\left[ W^2(t)^2 \right] - 2t^2 + t^2 \\
= 2t^2.
\]
Hence, indeed, \( \mathbb{E}Q_{1/2} = \frac{T}{2} \) and \( \lim_{||\Pi|| \to 0} \text{Var}(Q_{1/2}) = 0 \), so that
\[
\lim_{||\Pi|| \to 0} Q_{1/2} = \frac{T}{2},
\]
which is the required result.

(ii) Define the Stratonovich integral of \( W(t) \) with respect to \( W(t) \) to be
\[
\int_0^T W(t) \circ dW(t) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} W(t_j^*) (W(t_{j+1}) - W(t_j)). 
\] (4.10.1)
In contrast to the Itô integral \( \int_0^T W(t) dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T \) of (4.3.4), which evaluates the integrand at the left endpoint of each subinterval \( [t_j, t_{j+1}] \), here we evaluate the integrand at the midpoint \( t_j^* \). Show that
\[
\int_0^T W(t) \circ dW(t) = \frac{1}{2}W^2(T).
\]
(Hint: Write the approximating sum in (4.10.1) as the sum of an approximating sum for the Itô integral \( \int_0^T W(t) dW(t) \) and \( Q_{1/2} \). The approximating sum for the Itô integral is the one corresponding to the partition \( 0 = t_0 < t_1 < t_2^* < \cdots < t_{n-1}^* < t_n = T \), not the partition \( \Pi \).

First, notice that
\[
a(b - c) = [a(b - a) + c(a - c)] + (a - c)^2,
\]
so that we can express the term under the limit in (4.10.1) as
\[
\sum_{j=0}^{n-1} W(t_j^*) (W(t_{j+1}) - W(t_j)) \\
= \sum_{j=0}^{n-1} \left[ W(t_j^*) (W(t_{j+1}) - W(t_j^*)) + W(t_j^*) (W(t_j^*) - W(t_j)) \right] + \sum_{j=0}^{n-1} (W(t_j^*) - W(t_j))^2
\]
\[
\leq \frac{T^2}{2} \int_0^T W_t dW_t + Q_{1/2}.
\]
Hence, the partial sum which converges in $L^2$ to the Stratonovich integral can be expresses as a sum of two terms. The first of them converges in $L^2$ to $\int_0^T W(t)dW(t)$, i.e. the Itô integral of the Brownian motion, which we know\(^4\) is equal to

$$\int_0^T W(t)dW(t) = \frac{W^2(T)}{2} - \frac{T}{2},$$

This is because this term boils down to a sum over a finer partition $\Pi^*$, with $2n$ elements, created by augmenting the old partition $\Pi$ by putting $t^*_j$'s between each $t_j$ and $t_{j+1}$, i.e.

$$\Pi^* = \{0 = t_0, t_0^*, t_1^*, \ldots, t_{n-1}, t^*_n = t_n = T\},$$

so that

$$\sum_{j=0}^{n-1} \left[ W(t^*_j) (W(t_{j+1}) - W(t^*_j)) + W(t_j) (W(t^*_j) - W(t_j)) \right] = \sum_{k=0}^{2n-1} W(t_k) (W(t_{k+1}) - W(t_k)),$$

where

$$t_k = \begin{cases} t_j, & \text{if } 2 \mid k, \\ t^*_j, & \text{if } 2 \nmid k. \end{cases}$$

with the RHS in (2) is indeed the term under the limit in the Itô integral of the Brownian motion.

The second term is the half-sample quadratic variation we considered in previous point, so we already know that in the limit it goes to $\frac{T}{2}$.

Summing up, we can state that

$$\sum_{j=0}^{n-1} W(t^*_j) (W(t_{j+1}) - W(t_j)) \overset{L^2}{\rightarrow} \int_0^T W(t)dW(t) + \frac{T}{2} + \frac{T}{2} = \frac{W^2(T)}{2},$$

which completes the proof.

\(^4\)Cf. (4.3.6) in Ex. 4.3.2. in Shreve.