Measure theory and stochastic processes
TA Session Problems No. 5

Agnieszka Borowska

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Note: this is only a draft of the solutions discussed on Wednesday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

Ex. 4.1 (Shreve)

Suppose $M(t)$, $0 \leq t \leq T$ is a martingale with respect to some filtration $\mathcal{F}(t)$, $0 \leq t \leq T$. Let $\Delta(t)$, $0 \leq t \leq T$, be a simple process adapted to $\mathcal{F}(t)$ (i.e., there is a partition $\Pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$ such that, for every j, $\Delta(t_j)$ is $\mathcal{F}(t_j)$-measurable and $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$. For $t \in [t_k, t_{t+1})$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j) \left[M(t_{j+1}) - M(t_j)\right] + \Delta(t_k) \left[M(t) - M(t_k)\right]. \tag{1}$$

We think of $M(t)$ as the price of an asset at time t and $\Delta(t_j)$ as the number of shares of the asset held by an investor between times t_j and t_{j+1}. Then $I(t)$ is the capital gains that accrue to the investor between times 0 and t. Show that $I(t)$, $0 \leq t \leq T$, is a martingale.

First, recall the definition of a **martingale**.

Def 2.3.5(i). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathcal{F}$. Consider an adapted stochastic process $M(t)$, $0 \leq t \leq T$. If*

$$\mathbb{E}\left[M(t) | \mathcal{F}(s)\right] = M(s), \quad \text{for all } 0 \leq s \leq t \leq T,$$

we say this process is a martingale. It has no tendency to rise or fall.

Next, recall two important notions. Below we let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$, where $T > 0$ is fixed, i.e. $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$.

A **simple process** $\Delta(t)$ is an adapted stochastic process, which is constant in t on each subinterval $[t_j, t_{j+1})$.

The **Itô integral of a simple process** $\Delta(t)$ is is a stochastic process given by

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j) \left[W(t_{j+1}) - W(t_j))\right] + \Delta(t_k) \left[W(t) - W(t_k))\right], \tag{4.2.2}$$

where $t_k \leq t \leq t_{k+1}$, which is denoted as

$$I(t) = \int_0^t \Delta(u) dW(u).$$

Finally, recall that the Itô integral is a **martingale**.

Thm. 4.3.1. *The Itô integral defined by (4.2.2) is a martingale.*

Let $0 \le s \le t \le T$ and wlog[1] assume $s = t_l$ and $t = t_k$, for some l, k. We need to check what the expectations of $I(t)$ (1) given $\mathcal{F}(s)$ is. We have

$$\begin{aligned}
\mathbb{E}\left[I(t)|\mathcal{F}(s)\right] &= \mathbb{E}\left[I(t_k)|\mathcal{F}(t_l)\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{k-1} \Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] \middle| \mathcal{F}(t_l)\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{l-1} \Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] + \sum_{j=l}^{k-1} \Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] \middle| \mathcal{F}(t_l)\right] \\
&\overset{\text{lin.}}{=} \sum_{j=0}^{l-1} \mathbb{E}\Big[\Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] \big|\mathcal{F}(t_l)\Big] + \sum_{j=l}^{k-1} \mathbb{E}\Big[\Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] \big|\mathcal{F}(t_l)\Big] \\
&\overset{\text{measur.}}{\underset{\text{IC}}{=}} \sum_{j=0}^{l-1} \Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] + \sum_{j=l}^{k-1} \mathbb{E}\Big[\, \mathbb{E}\left[\Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] \big|\mathcal{F}(t_j)\right] \big|\, \mathcal{F}(t_l)\Big] \\
&= I(s) + \sum_{j=l}^{k-1} \mathbb{E}\Big[\, \mathbb{E}\left[\Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] \big|\mathcal{F}(t_j)\right] \big|\, \mathcal{F}(t_l)\Big] \\
&\overset{\text{M-marting.}}{\underset{\text{measur.}}{=}} I(s) + \sum_{j=l}^{k-1} \mathbb{E}\Big[\Delta(t_j)\left[M(t_j) - M(t_j)\right] \big|\mathcal{F}(t_l)\Big] \\
&= I(s) + \sum_{j=l}^{k-1} \mathbb{E}\Big[0\big|\mathcal{F}(t_l)\Big] \\
&= I(s) + 0 \\
&= I(s),
\end{aligned}$$

where IC denotes iterated conditioning[2], which shows that $I(t)$ is a martingale.

Notice that the "trick" with iterated conditioning allowed us to make use of the martigale property of the process M, i.e. we could write that

$$\begin{aligned}
\mathbb{E}\Big[\Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] \big|\mathcal{F}(t_j)\Big] &= \Delta(t_j)\mathbb{E}\left[M(t_{j+1})\big|\mathcal{F}(t_j)\right] - \Delta(t_j)M(t_j) \\
&= \Delta(t_j)M(t_j) - \Delta(t_j)M(t_j) \\
&= 0.
\end{aligned}$$

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[1]Without loss of generality. Indeed, as we can always take a new partition of $[0, T]$ with re-arranged indices.

[2]Cf. Thm. 2.3.2(iii): *If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$ ($\mathcal{H}$ contains less information than $\mathcal{G}$) and X is an integrable random variable, then*

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]|\mathcal{H}\right] = \mathbb{E}\left[X|\mathcal{H}\right]. \qquad (2.3.20)$$

Before moving to the next exercesie, let us go through a short recap on convergence and general Itô integrals.

First, recall the definition of **convergence in the p-th moment.**

Def. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables and X be a random variable defined on the same probability space. We say that $(X_n)_{n=1}^{\infty}$ converges to X in the p-th moment (in L^p), $0 < p < \infty$, if $\mathbb{E}|X|^p < \infty$, $\mathbb{E}|X_n|^p < \infty$, $\forall n$, and*

$$\lim_{n\to\infty} \mathbb{E}[X_n - X]^p = 0,$$

and we denote this by $X_n \overset{L^p}{\to} X$.

Second, recall the **Itô isometry** property of the Itô integral (4.2.2).

Thm. 4.2.2. *The Itô integral defined by* (4.2.2) *satisfies*

$$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du. \tag{4.2.6.}$$

Formula (4.2.6.) allows us to compute $\text{Var}I(t) = \mathbb{E}I^2(t)$, where the latter equality follows from the fact that $\mathbb{E}I(t) = 0$, $\forall t \geq 0$.

Next, for a general integrand, being an adapted stochastic process $\Delta(t)$, its Itô integral is constructed by approximating $\Delta(t)$ by simple processes $\Delta_n(t)$. The latter are chosen in such a way that they *converge* to the continuously varying $\Delta(t)$, which means that

$$\lim_{t\to\infty} \mathbb{E}\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0. \tag{4.3.2}$$

More formaly, the **Itô integral for the continuously varying integrand** $\Delta(t)$ is defined by the formula

$$I(t) = \int_0^t \Delta(u)dW(u) := \lim_{n\to\infty}\int_0^t \Delta_n(u)dW(u), \qquad 0 \leq t \leq T. \tag{4.3.3}$$

For each t, the limit in (4.3.3) exists because $I_n(t) = \int_0^t \Delta_n(u)dW(u)$ is a *Cauchy sequence* in $L^2(\Omega, \mathcal{F}, \mathbb{P})$[3]. This is because of Itô's isometry (Thm. 4.2.2), which yields

$$\mathbb{E}\left(I_n(t) - I_m(t)\right)^2 = \mathbb{E}\int_0^t \left|\Delta_n(u) - \Delta_m(u)\right|^2 du.$$

As a consequence of (4.3.2), the right-hand side has limit zero as n and m approach infinity.

Finally, recall the **properties of the Itô integral**.

Thm. 4.3.1. *Let T be a positive constant and let $\Delta(t)$, $0 \leq t \leq T$, be an adapted stochastic process that satisfies*

$$\mathbb{E}\int_0^T \Delta^2(t)dt < \infty. \tag{4.3.1}$$

Then $I(t) = \int_0^t \Delta(u)dW(u)$ defined by (4.3.3) *has the following properties.*

(a) ***(Continuity)*** *As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*

(b) ***(Adaptivity)*** *For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.*

(c) ***(Linearity)*** *If $I(t) = \int_0^t \Delta(u)dW(u)$ and $J(t) = \int_0^t \Gamma(u)dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u))\, dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)dW(u)$.*

(d) ***(Martingale)*** *$I(t)$ is a martingale.*

(e) ***(Itô isometry)*** *$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du$.*

(f) ***(Quadratic variation)*** *$[I, I](t) = \int_0^t \Delta^2(u)du$.*

[3]For $0 \leq \infty$, the L^p spaces are *complete* (when equipped with an appropriate norm).

Ex. 4.4 (Shreve) (Stratonovich integral)

Let $W(t)$, $t \geq 0$, be a Brownian motion. Let T be a fixed positive number and let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0,T]$ (i.e., $0 = t_0 < t_1 < \dots < t_n = T$). For each j, define $t_j^* = \frac{t_j + t_{j+1}}{2}$ to be the midpoint of the interval $[t_j, t_{j+1}]$.

(i) *Define the* half-sample quadratic variation *corresponding to Π to be*

$$Q_{\Pi/2} = \sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j)\right)^2.$$

Show that $Q_{\Pi/2}$ has limit $\frac{1}{2}T$ as $||\Pi|| \to 0$.
(Hint: It suffices to show that $\mathbb{E}Q_{\Pi/2} = \frac{1}{2}T$ and $\lim_{||\Pi||\to 0} Var(Q_{\Pi/2}) = 0$.)

In this exercise we will consider convergence in L^2, since the Stratonovich integral is defined as the limit in L^2 (similarly to the Itô integral).

Using the hint we can start with computing the expected value of the half-sample quadratic variation under consideration. We have

$$\begin{aligned}
\mathbb{E}(Q_{\Pi/2}) &= \mathbb{E}\left[\sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j)\right)^2\right] \\
&= \sum_{j=0}^{n-1} \mathbb{E}\left[\left(W(t_j^*) - W(t_j)\right)^2\right] \\
&\overset{(*)}{=} \sum_{j=0}^{n-1} \left(t_j^* - t_j\right) \\
&= \sum_{j=0}^{n-1} \left(\frac{t_j + t_{j+1}}{2} - t_j\right) \\
&= \sum_{j=0}^{n-1} \frac{t_{j+1} - t_j}{2} \\
&= \frac{T}{2},
\end{aligned}$$

where in $(*)$ we use the fact that for $0 \leq s \leq t$ the increment of the Brownian motion $W(t) - W(s) \sim N(0, t-s)$. Notice that in the last step we used the following equality (we will use it in the next point)

$$\frac{T}{2} = \sum_{j=0}^{n-1} \frac{t_{j+1} - t_j}{2}.$$

Next, for the variance we have

$$
\begin{aligned}
\operatorname{Var}(Q_{\Pi/2}) =& \mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)^2-\frac{T}{2}\right)^2\right] \\
=& \mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)^2-\sum_{j=0}^{n-1}\frac{t_{j+1}-t_j}{2}\right)^2\right] \\
=& \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\mathbb{E}\left[\left(\left(W(t_j^*)-W(t_j)\right)^2-\frac{t_{j+1}-t_j}{2}\right)\left((W(t_k^*)-W(t_k))^2-\frac{t_{k+1}-t_k}{2}\right)\right] \\
=& \sum_{j=0}^{n-1}\mathbb{E}\left[\left(W(\tilde{t}_j)^2-\frac{t_{j+1}-t_j}{2}\right)^2\right] \\
\overset{(**)}{=}& \sum_{j=0}^{n-1}2\cdot\left(\frac{t_{j+1}-t_j}{2}\right)^2 \\
\leq& \frac{T}{2}\max_{1\leq j\leq n}|t_{j+1}-t_j| \\
\to& \, 0,
\end{aligned}
$$

where $\tilde{t}_j=\frac{t_{j+1}-t_j}{2}$ and $(**)$ is because of

$$
\begin{aligned}
\mathbb{E}\left[(W^2(t)-t)^2\right] &= \mathbb{E}\left[W^4(t)-2tW^2(t)+t^2\right] \\
&= \mathbb{E}\left[W^2(t)\right]^2-2t^2+t^2 \\
&= 2t^2.
\end{aligned}
$$

Hence, indeed, $\mathbb{E}Q_{\Pi/2}=\frac{T}{2}$ and $\lim_{||\Pi||\to 0}\operatorname{Var}(Q_{\Pi/2})=0$, so that

$$\lim_{||\Pi||\to 0} Q_{\Pi/2}=\frac{T}{2},$$

which is the required result.

(ii) *Define the Stratonovich integral of $W(t)$ with respect to $W(t)$ to be*

$$\int_0^T W(t)\circ dW(t)=\lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}W(t_j^*)\left(W(t_{j+1})-W(t_j)\right). \tag{4.10.1}$$

In contrast to the Itô integral $\int_0^T W(t)dW(t)=\frac{1}{2}W^2(T)-\frac{1}{2}T$ of (4.3.4), which evaluates the integrand at the left endpoint of each subinterval $[t_j,t_{j+1}]$, here we evaluate the integrand at the midpoint t_j^*. Show that

$$\int_0^T W(t)\circ dW(t)=\frac{1}{2}W^2(T).$$

(Hint: Write the approximating sum in (4.10.1) as the sum of an approximating sum for the Itô integral $\int_0^T W(t)dW(t)$ and $Q_{\Pi/2}$. The approximating sum for the Ito integral is the one corresponding to the partition $0=t_0<t_0^*<t_1<t_1^*<\cdots<t_{n-1}^*<t_n=T$, not the partition Π.)

First, notice that

$$a(b-c)=[a(b-a)+c(a-c)]+(a-c)^2,$$

so that we can express the term under the limit in (4.10.1) as

$$
\begin{aligned}
&\sum_{j=0}^{n-1}W(t_j^*)\left(W(t_{j+1})-W(t_j)\right) \\
=&\underbrace{\sum_{j=0}^{n-1}\left[W(t_j^*)\left(W(t_{j+1})-W(t_j^*)\right)+W(t_j)\left(W(t_j^*)-W(t_j)\right)\right]}_{\overset{L^2}{\to}\int_0^T W_t dW_t}+\underbrace{\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)^2}_{=Q_{\Pi/2}}.
\end{aligned}
$$

Hence, the partial sum which converges in L^2 to the Stratonovich integral can be expresses as a sum of two terms. The first of them converges in L^2 to $\int_0^T W(t)dW(t)$, i.e. the Itô integral of the Brownian motion, which we know[4] is equal to

$$\int_0^T W(t)dW(t) = \frac{W^2(T)}{2} - \frac{T}{2}.$$

This is because this term boils down to a sum over a finer partition Π^*, with $2n$ elements, created by augmenting the old partition Π by putting t_j^*'s between each t_j and t_{j+1}, i.e.

$$\Pi^* = \{0 = t_0, t_0^*, t_1, t_1^*, \ldots, t_{n-1}, t_{n-1}^*, t_n = T\},$$

so that

$$\sum_{j=0}^{n-1} \left[W(t_j^*)\left(W(t_{j+1}) - W(t_j^*)\right) + W(t_j)\left(W(t_j^*) - W(t_j)\right)\right] = \sum_{k=0}^{2n-1} W(t_k)\left(W(t_{k+1}) - W(t_k)\right), \tag{2}$$

where

$$t_k = \begin{cases} t_j,\ j = \frac{k}{2} & \text{if } 2 \mid k, \\ t_j^*,\ j = \frac{k-1}{2}, & \text{if } 2 \not\mid k. \end{cases}$$

with the RHS in (2) is indeed the term under the limit in the Itô integral of the Brownian motion.

The second term is the half-sample quadratic variation we considered in previous point, so we already know that in the limit it goes to $\frac{T}{2}$.

Summing up, we can state that

$$\begin{aligned} &\sum_{j=0}^{n-1} W(t_j^*)\left(W(t_{j+1}) - W(t_j)\right) \\ &\xrightarrow{L^2} \int_0^T W(t)dW(t) + \frac{T}{2} \\ &= \frac{W^2(T)}{2} - \frac{T}{2} + \frac{T}{2}, \\ &= \frac{W^2(T)}{2}, \end{aligned}$$

which completes the proof.

[4] Cf. (4.3.6) in Ex. 4.3.2. in Shreve.