Measure theory and stochastic processes
TA Session Problems No. 4

Agnieszka Borowska

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Note: this is only a draft of the solutions discussed on Friday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

Ex. 3.1 (Shreve)

According to Definition 3.3.3(iii), for $0 \leq t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the σ-algebra $\mathcal{F}(t)$. Use this property and property (i) of that definition to show that, for $0 \leq t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

First, for completeness, recall the definition of a **Brownian motion.**

Def 3.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $W(t)$, $t \geq 0$, is a Brownian motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments*

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1}) \tag{3.3.1}$$

are independent and each of these increments is normally

$$\mathbb{E}\left[W(t_{i+1}) - W(t_i)\right] = 0, \tag{3.3.2}$$
$$Var\left[W(t_{i+1}) - W(t_i)\right] = t_{i+1} - t_i. \tag{3.3.3}$$

Second, recall the definition of a **filtration** and a **filtration for the Brownian motion.**

Def 2.1.1. *Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}(t)$. Assume further that if $s \leq t$, then every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. Then we call the collection of σ-algebras $\mathcal{F}(t)$, $0 \leq t \leq T$ a filtration.*

Def 3.3.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t)$, $t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathcal{F}(t)$, $t \geq 0$, satisfying*

- *(i)* ***(Information accumulates)*** *For $0 \leq s < t$ every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. In other words, there is at least as much information available at the later time $\mathcal{F}(t)$ as there is at the earlier time $\mathcal{F}(s)$.*
- *(ii)* ***(Adaptivity)*** *For each $t \geq 0$, the Brownian motion $W(t)$ at time t is $\mathcal{F}(t)$-measurable. In other words, the information available at time t is sufficient to evaluate the Brownian motion $W(t)$ at that time.*
- *(iii)* ***(Independence of future increments)*** *For $0 \leq t < u$, the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$. In other words, any increment of the Brownian motion after time t is independent of the information available at time t.*

Let $\Delta(t)$, $t \geq 0$ be a stochastic process. We say that $\Delta(t)$ is adapted to the filtration $\mathcal{F}(t)$ if for each $t \geq 0$ the random variable $\Delta(t)$ is $\mathcal{F}(t)$-measurable.

Since $0 \leq t < u_1 < u_2$ and $\mathcal{F}(t)$ is a filtration, we have $\mathcal{F}(t) \subset \mathcal{F}(u_1) \subset \mathcal{F}(u_2)$. Next, since W is a Brownian motion, the increment $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(u_1)$. Hence, in particular, this increment is independent of $\mathcal{F}(t)$.

Ex. 3.4 (Shreve) (Other variations of Brownian motion)

Theorem 3.4.3 asserts that if T is a positive number and we choose a partition Π with points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, then as the number n of partition points approaches infinity and the length of the longest subinterval $||\Pi||$ approaches zero, the sample quadratic variation

$$\sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

approaches T for almost every path of the Brownian motion W. In Remark 3.4.5, we further showed that $\sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)(t_{j+1} - t_j)$ and $\sum_{j=0}^{n-1} (t_{j+1} - t_j)^2$ have limit zero. We summarize these facts by the multiplication rules

$$dW(t)dW(t) = dt, \quad dW(t)dt = 0, \quad dtdt = 0. \tag{3.10.1}$$

(i) Show that as the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample first variation

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

approaches ∞ for almost every path of the Brownian motion W. (Hint:

$$\sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2 \le \max_{0 \le k \le n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|.)$$

To start with, recall the notion of a **quadratic variation** and theorem 3.4.3.

Def. 3.4.1. *Let $f(t)$ be a function defined for $0 \le t \le T$. The quadratic variation of f up to time T is*

$$[f, f](T) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \left[f(t_{j+1}) - f(t_j)\right]^2, \tag{3.4.5}$$

where $\Pi = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$.

Thm. 3.4.3. *Let W be a Brownian motion. Then $[W, W](T) = T$ for all $T \ge 0$ almost surely.*

Next, by contradiction, suppose that there exists a set $A \in \mathcal{F}$, with a positive probability $\mathbb{P}(A) > 0$, such that $\forall \omega \in A$ the sample first variation is finite:

$$\sum_{j=0}^{n-1} |W_{t_{j+1}}(\omega) - W_{t_j}(\omega)| < \infty. \tag{1}$$

Then, $\forall \omega \in A$,

$$\underbrace{\sum_{j=0}^{n-1} \left(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right)^2}_{\to T} \le \underbrace{\max_{0 \le k \le n-1} \left|W_{t_{k+1}}(\omega) - W_{t_k}(\omega)\right|}_{\to 0} \cdot \underbrace{\sum_{j=0}^{n-1} \left|W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right|}_{<\infty} \to 0,$$

which is a contradiction. The first brace follows from theorem 3.4.3: for a Brownian motion

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} \left(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right)^2 = T, \quad \text{a.s.};$$

the second brace is due to the assumption that the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, i.e.

$$\lim_{n \to \infty} \max_{0 \le k \le n-1} \left|W_{t_{k+1}}(\omega) - W_{t_k}(\omega)\right| = 0,$$

the last brace is because our assumption (1).

Hence, $\mathbb{P}(A) = 0$, or, in other words, the sample first variation is a.s. infinite.

(ii) Show that as the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample cubic variation

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3$$

approaches zero for almost every path of the Brownian motion W.

We can proceed similarly as in the previous point - just notice that

$$\sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j}\right)^3 \leq \underbrace{\max_{0\leq k\leq n-1} \left|W_{t_{k+1}} - W_{t_k}\right|}_{\to 0} \cdot \underbrace{\sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j}\right)^2}_{\to T} \to 0.$$

Ex. 3.6 (Shreve)

Let $W(t)$ be a Brownian motion and let $\mathcal{F}(t)$, $t \geq 0$ be an associated filtration.

(i) For $\mu \in \mathbb{R}$ consider the Brownian motion with drift μ:

$$X(t) = \mu t + W(t). \tag{2}$$

Show that for any Borel-measurable function $f(y)$, and for any $0 \leq s < t$, the function

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{(y-x-\mu(t-s))^2}{2(t-s)}\right\} dy \tag{3}$$

satisfies $\mathbb{E}\left[f(X(t))|\mathcal{F}(s)\right] = g(X(s))$, and hence X has the Markov property. We may rewrite $g(x)$ as $g(x) = \int_{-\infty}^{\infty} f(y)p(\tau, x, y)dy$, where $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{-\frac{(y-x-\mu\tau)^2}{2\tau}\right\}$$

is the transition density for Brownian motion with drift μ.

Recall the definition of a **Markov process**.

Def. 2.3.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathcal{F}(t)$, $0 \leq t \leq T$ be a filtration of sub-σ-algebras of $\mathcal{F}$. Consider an adapted stochastic process $X(t)$, $0 \leq t \leq T$. Assume that for all $0 \leq s < t \leq T$ and for every nonnegative, Borel-measurable function f, there is another Borel-measurable function g such that*

$$\mathbb{E}\left[f(X(t))|\mathcal{F}(s)\right] = g(X(s)). \tag{2.3.29}$$

Then we say that the X is a Markov process.

We can write

$$\mathbb{E}\left[f(X(t))|\mathcal{F}(s)\right] = \mathbb{E}\left[f(X(t) - X(s) + X(s))|\mathcal{F}(s)\right]$$

and define

$$h(x) = \mathbb{E}\left[f(X(t) - X(s) + x)|\mathcal{F}(s)\right],$$

so that we will have

$$h(X(s)) = \mathbb{E}\left[f(X(t) - X(s) + X(s))|\mathcal{F}(s)\right].$$

Since W is a Brownian motion, $W(t) - W(s) \sim N(0, t-s)$, and because we have defined X as a Brownian motion with a drift (2), we have

$$\begin{aligned} X(t) - X(s) &= (\mu t + W(t)) - (\mu s + W(s)) \\ &= \mu(t-s) + W(t) - W(s) \\ &\sim N\left(\mu(t-s), t-s\right). \end{aligned}$$

Then, we can calculate $h(x)$ as the expected value of a function f of a normal variable $X(t) - X(s)$ plus x[1]:

$$\begin{aligned} h(x) &= \mathbb{E}\left[f(X(t) - X(s) + x)|\mathcal{F}(s)\right] \\ &\overset{(*)}{=} \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(u+x) \exp\left\{-\frac{(u-\mu(t-s))^2}{2(t-s)}\right\} du \\ &\underset{du=dy}{\overset{y:=u+x}{=}} \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{(y-x-\mu(t-s))^2}{2(t-s)}\right\} dy. \end{aligned}$$

Therefore,

$$\begin{aligned} h(X(s)) &= \mathbb{E}\left[f(X(t) - X(s) + X(s))|\mathcal{F}(s)\right] \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{(y-X(s)-\mu(t-s))^2}{2(t-s)}\right\} dy \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) p(t-s, X_s, y) dy, \end{aligned}$$

where

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{-\frac{(y-x-\mu\tau)^2}{2\tau}\right\}.$$

So it turns out that if we take $h = g$, with g defined as (3), we have that $\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$, for $0 \leq s < t$. Hence, indeed, X has Markov property.

(ii) For $\nu \in \mathbb{R}$ and $\sigma > 0$, consider the geometric Brownian motion

$$S(t) = S(0)e^{\sigma W(t) + \nu t}. \tag{4}$$

Set $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sigma y \sqrt{2\pi\tau}} \exp\left\{-\frac{(\log\frac{y}{x} - \nu\tau)^2}{2\sigma^2\tau}\right\}.$$

how that for any Borel-measurable function $f(y)$, and for any $0 \leq s < t$, the function

$$g(x) = \int_0^{\infty} f(y) p(\tau, x, y) dy \tag{5}$$

satisfies $\mathbb{E}\left[f(S(t))|\mathcal{F}(s)\right] = g(S(s))$, and hence S has the Markov property.

Now, we write

$$\mathbb{E}\left[f(S(t))|\mathcal{F}(s)\right] = \mathbb{E}\left[f\left(\frac{S(t)}{S(s)} \cdot S(s)\right)\middle|\mathcal{F}(s)\right]$$

and define

$$h(x) = \mathbb{E}\left[f\left(\frac{S(t)}{S(s)} \cdot x\right)\middle|\mathcal{F}(s)\right],$$

so that we will have

$$h(S(s)) = \mathbb{E}\left[f\left(\frac{S(t)}{S(s)} \cdot S(s)\right)\middle|\mathcal{F}(s)\right].$$

Since W is a Brownian motion, $W(t) - W(s) \sim N(0, t-s)$, and because we have defined S as a geometric Brownian motion (4), we have

$$\log\frac{S(t)}{S(s)} = \sigma\big(W(t) - W(s)\big) + \nu(t-s) \sim N\big(\nu(t-s), \sigma^2(t-s)\big).$$

[1] Recall the independence property of conditional expectation (thm. 2.3.2(iv)), which states that if X is integrable and independent of $\mathcal{G}$, where $\mathcal{G} \subset \mathcal{F}$, then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X. \tag{2.3.21}$$

We already know that for $0 \leq s < t$ the increment of a Brownian motion $W(t) - W(s)$ is independent of $\mathcal{F}(s)$, so in our case the conditional expectation given $\mathcal{F}(s)$, i.e. $\mathbb{E}\left[f(X(t) - X(s) + x)|\mathcal{F}(s)\right]$, is just the ordinary expectation $\mathbb{E}\left[f(X(t) - X(s) + x)\right]$. The latter can be calculated using the well known formula involving the density of a normal variable. Recall: if $Y \sim N(\mu, \sigma^2)$, then it has the density function $f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(u-\mu)^2}{2\sigma^2}\right\}$ so that for any Borel measurable function φ, we have $\mathbb{E}[\varphi(Y)] = \int_{-\infty}^{\infty} \varphi(u) f(u) du$. So, for some constant $b \in \mathbb{R}$, we have $\mathbb{E}[\varphi(Y + b)] = \int_{-\infty}^{\infty} \varphi(u+b) f(u) du$. This is exactly what we apply in $(*)$.

Then, we can calculate $h(x)$ as the expected value of a function f of an exponent of a normal variable $\log\frac{S(t)}{S(s)}$ times x (for conditioning cf. footnote 1):

$$
\begin{aligned}
h(x) &= \mathbb{E}\left[f\left(\frac{S(t)}{S(s)}\cdot x\right) \middle| \mathcal{F}(s)\right] \\
&= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \int_{-\infty}^{\infty} f(e^u x) \exp\left\{ -\frac{(u-\nu(t-s))^2}{2\sigma^2(t-s)} \right\} du \\
&\underset{du=dy/y}{\overset{y:=e^u x}{=}} \frac{1}{\sqrt{2\sigma^2\pi(t-s)}} \int_{0}^{\infty} \frac{f(y)}{y} \exp\left\{ -\frac{\left(\log\frac{y}{x}-\nu(t-s)\right)^2}{2\sigma^2(t-s)} \right\} dy,
\end{aligned}
$$

where the limits of the last integral are due to $\lim_{u\to\infty} e^u x = \infty$ and $\lim_{u\to-\infty} e^u x = 0$. Therefore,

$$
\begin{aligned}
h(S(s)) &= \mathbb{E}\left[f\left(\frac{S(t)}{S(s)}\cdot S(s)\right) \middle| \mathcal{F}(s)\right] \\
&= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \int_{0}^{\infty} \frac{f(y)}{y} \exp\left\{ -\frac{\left(\log\frac{y}{S(s)}-\nu(t-s)\right)^2}{2\sigma^2(t-s)} \right\} dy. \\
&= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \int_{-\infty}^{\infty} f(y) p(t-s, S(s), y) dy,
\end{aligned}
$$

where

$$
p(\tau, x, y) = \frac{1}{y\sqrt{2\pi\sigma^2\tau}} \exp\left\{ -\frac{\left(\log\frac{y}{x}-\nu\tau\right)^2}{2\sigma^2\tau} \right\}.
$$

If we take $h = g$, with g defined as (5), we have that $\mathbb{E}[f(S(t))|\mathcal{F}(s)] = g(S(s))$, for $0 \le s < t$, which means that S has Markov property.