Measure theory and stochastic processes
TA Session Problems No. 3

Agnieszka Borowska

17.09.2014

Note: this is only a draft of the solutions discussed on Wednesday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

Ex. 2.8 (Shreve)

Independence of random variables can be affected by changes of measure. To illustrate this point, consider the space of two coin tosses $\Omega_2 = \{HH, HT, TH, TT\}$, and let stock prices be given by

$$S_0 = 4, \qquad S_1(H) = 8, \qquad S_1(T) = 2,$$
$$S_2(HH) = 16, \qquad S_2(HT) = S_2(TH) = 4, \qquad S_2(TT) = 1.$$

Consider two probability measures given by

$$\tilde{\mathbb{P}}\{HH\} = \frac{1}{4}, \qquad \tilde{\mathbb{P}}\{HT\} = \frac{1}{4}, \qquad \tilde{\mathbb{P}}\{TH\} = \frac{1}{4}, \qquad \tilde{\mathbb{P}}\{TT\} = \frac{1}{4},$$
$$\mathbb{P}\{HH\} = \frac{4}{9}, \qquad \mathbb{P}\{HT\} = \frac{2}{9}, \qquad \mathbb{P}\{TH\} = \frac{2}{9}, \qquad \mathbb{P}\{TT\} = \frac{1}{9}.$$

Define the random variable

$$X = \begin{cases} 1 & \text{if } S_2 = 4, \\ 0 & \text{if } S_2 \neq 4. \end{cases}$$

(i) *List all the sets in $\sigma(X)$.*

First, recall the definition of a σ-algebra generated by a random variable.

Def. 2.1.3. *Let X be a random variable defined on a sample space $\Omega \neq \emptyset$. The σ-algebra generated by X, denoted $\sigma(X)$ is the collection of all subsets of Ω the form $\{X \in C\}$, where C ranges over the Borel subsets of $\mathbb{R}$.*

So, we have

$$\begin{aligned} \sigma(X) &= \{\emptyset, \Omega, \{X = 1\}, \{X = 0\}\} \\ &= \{\emptyset, \Omega, \{S_2 = 4\}, \{S_2 \neq 4\}\} \\ &= \{\emptyset, \Omega, \{HT, TH\}, \{HH, TT\}\}. \end{aligned}$$

(ii) *List all the sets in $\sigma(S_1)$.*

$$\begin{aligned} \sigma(S_1) &= \{\emptyset, \Omega, \{S_1 = 8\}, \{S_1 = 2\}\} \\ &= \{\emptyset, \Omega, \{\omega \in \Omega_2 : S_1 = 8\}, \{\omega \in \Omega_2 : S_1 = 2\}\} \\ &= \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\}. \end{aligned}$$

(iii) *Show that $\sigma(X)$ and $\sigma(S_1)$ are independent under the probability measure $\tilde{\mathbb{P}}$.*

Now, recall the definition of independent σ-algebras and independent random variables.

Def. 2.2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$ (i.e., the sets in $\mathcal{G}$ and the sets in $\mathcal{H}$ are also in $\mathcal{F}$). We say these two σ-algebras are independent if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B), \quad \forall A \in \mathcal{G}, B \in \mathcal{H}. \tag{1}$$

Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We say these two random variables are independent if the σ-algebras they generate, $\sigma(X)$ and $\sigma(Y)$, are independent. We say that the random variable X is independent of the σ-algebra $\mathcal{G}$ if $\sigma(X)$ and $\mathcal{G}$ are independent.

Hence, we need to check whether $\forall A \in \sigma(X)$ and $\forall B \in \sigma(S_1)$ the probability (given the appropriate probability measure) of an intersection $A \cap B$ factorises according to (1). Let us consider only the non-trivial cases (i.e. the sets $A, B \notin \{\emptyset, \Omega\}$).

$A \in \sigma(X)$	$B \in \sigma(S_1)$	$A \cap B$	$\tilde{\mathbb{P}}(A) \cdot \tilde{\mathbb{P}}(B)$	$\tilde{\mathbb{P}}(A \cap B)$
$\{HT, TH\}$	$\{HH, HT\}$	$\{HT\}$	$(\frac{1}{4} + \frac{1}{4}) \cdot (\frac{1}{4} + \frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$
$\{HT, TH\}$	$\{TH, TT\}$	$\{TH\}$	$(\frac{1}{4} + \frac{1}{4}) \cdot (\frac{1}{4} + \frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$
$\{HH, TT\}$	$\{HH, HT\}$	$\{HH\}$	$(\frac{1}{4} + \frac{1}{4}) \cdot (\frac{1}{4} + \frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$
$\{HH, TT\}$	$\{TH, TT\}$	$\{TT\}$	$(\frac{1}{4} + \frac{1}{4}) \cdot (\frac{1}{4} + \frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$

So, indeed, $\forall A \in \sigma(X)$ and $\forall B \in \sigma(S_1)$ condition (1) holds, i.e. $\tilde{\mathbb{P}}(A \cap B) = \tilde{\mathbb{P}}(A) \cdot \tilde{\mathbb{P}}(B)$, which means that the σ-algebras $\sigma(X)$ and $\sigma(S_1)$ are independent under $\tilde{\mathbb{P}}$, and so are the random variables X and S_1.

(iv) *Show that $\sigma(X)$ and $\sigma(S_1)$ are not independent under the probability measure $\mathbb{P}$.*

Similarly as in the previous point, consider only the non-trivial cases.

$A \in \sigma(X)$	$B \in \sigma(S_1)$	$A \cap B$	$\mathbb{P}(A) \cdot \mathbb{P}(B)$	$\mathbb{P}(A \cap B)$
$\{HT, TH\}$	$\{HH, HT\}$	$\{HT\}$	$(\frac{2}{9} + \frac{2}{9}) \cdot (\frac{2}{9} + \frac{4}{9}) = \frac{8}{27}$	$\frac{2}{9}$
$\{HT, TH\}$	$\{TH, TT\}$	$\{TH\}$	$(\frac{2}{9} + \frac{2}{9}) \cdot (\frac{2}{9} + \frac{1}{9}) = \frac{4}{27}$	$\frac{2}{9}$
$\{HH, TT\}$	$\{HH, HT\}$	$\{HH\}$	$(\frac{4}{9} + \frac{1}{9}) \cdot (\frac{2}{9} + \frac{4}{9}) = \frac{10}{27}$	$\frac{4}{9}$
$\{HH, TT\}$	$\{TH, TT\}$	$\{TT\}$	$(\frac{4}{9} + \frac{1}{9}) \cdot (\frac{2}{9} + \frac{1}{9}) = \frac{1}{4}$	$\frac{1}{9}$

So, we can see that $\exists A \in \sigma(X)$ and $\exists B \in \sigma(S_1)$ such that $\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot (P)(B)$[1], which means that the σ-algebras $\sigma(X)$ and $\sigma(S_1)$ are not independent under $\mathbb{P}$, so neither are the random variables X and S_1.

(v) *Under $\mathbb{P}$, we have $\mathbb{P}\{S_1 = 8\} = \frac{2}{3}$ and $\mathbb{P}\{S_1 = 2\} = \frac{1}{3}$. Explain intuitively why, if you are told that $X = 1$, you would want to revise your estimate of the distribution of S_1.*

Since under $\mathbb{P}$ the random variables X and S_1 are *not* independent, the knowledge of the realisation of X is informative about the realisation of S_1, as it helps to revise our beliefs regarding the distribution of S_1. More precisely, if we know $X = 1$ then necessarily $S_2 = 4$, which means that either HT or TH has occured. And since $\mathbb{P}(HT) = \mathbb{P}(TH) = \frac{2}{9}$, we can update the initial beliefs about S_1 and estimate $\mathbb{P}\{S_1)$ using the formula for conditional probability

$$\mathbb{P}(S_1 = 8 | X = 1) = \frac{\mathbb{P}(S_1 = 8, X = 1)}{\mathbb{P}(X = 1)} = \frac{\frac{2}{9}}{\frac{4}{9}} = \frac{1}{2} = \mathbb{P}(S_1 = 2 | X = 1).$$

[1] Actually, we have shown that for no non-trivial subsets of Ω_2 condition (1) is satisfied.

Ex. 2.10 (Shreve)

Let X and Y be random variables (on some unspecified probability space $(\Omega, \mathcal{F}, \mathbb{P})$), assume they have a joint density $f_{X,Y}(x,y)$, and assume $\mathbb{E}|Y| < \infty$. In particular, for every Borel subset C of $\mathbb{R}$, we have

$$\mathbb{P}\{(X,Y) \in C\} = \int_C f_{X,Y}(x,y)dxdy.$$

In elementary probability, one learns to compute $\mathbb{E}[Y|X=x]$, which is a nonrandom *function of the* dummy variable *x, by the formula*

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy, \tag{2.6.1}$$

where $f_{Y|X}(y|x)$ is the conditional density *defined by*

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

The denominator in this expression, $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,\eta)d\eta$, is the marginal density *of X, and we must assume it is strictly positive for every x. We introduce the symbol $g(x)$ for the function $\mathbb{E}[Y|X=x]$ defined by* (2.6.1) *i.e.,*

$$g(x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_{-\infty}^{\infty} y \frac{f_{X,Y}(x,y)}{f_X(x)}dy.$$

In measure-theoretic probability, conditional expectation is a random variable *$\mathbb{E}[Y|X]$. This exercise is to show that when there is a joint density for (X,Y), this random variable can be obtained by substituting the random variable X in place of the dummy variable x in the function $g(x)$. In other words, this exercise is to show that*

$$\mathbb{E}[Y|X] = g(X).$$

(We introduced the symbol $g(x)$ in order to avoid the mathematically confusing expression $\mathbb{E}[Y|X=X]$.) Since $g(X)$ is obviously $\sigma(X)$-measurable, to verify that $\mathbb{E}[Y|X] = g(X)$ we need only check that the partial-averaging property is satisfied. For every Borel-measurable function h mapping $\mathbb{R}$ to $\mathbb{R}$ and satisfying $\mathbb{E}|h(X)| < \infty$, we have

$$\mathbb{E}h(X) = \int_{-\infty}^{\infty} h(x) f_X(x)dx. \tag{2.6.2}$$

This is Theorem 1.5.2 in Chapter 1. Similarly, if h is a function of both x and y, then

$$\mathbb{E}h(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f_{X,Y}(x,y)dxdy \tag{2.6.3}$$

whenever (X,Y) has a joint density $f_{X,Y}(x,y)$. You may use both (2.6.2) *and* (2.6.3) *in your solution to this problem.*
Let A be a set in $\sigma(X)$. By the definition of $\sigma(X)$ there is a Borel subset B of $\mathbb{R}$ such that $A = \{\omega \in \Omega : X(\omega) \in B\}$ or, more simply, $A = \{X \in B\}$. Show the partial-averaging property

$$\int_A g(X)d\mathbb{P} = \int_A Y d\mathbb{P}.$$

For sake of completeness, recall the definition of conditional expectation.

Def. 2.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a random variable that is either nonnegative or integrable. The* conditional expectation *of X given $\mathcal{G}$, denoted $\mathbb{E}[X|\mathcal{G}]$, is any random variable that satisfies*

(i) ***(Measurability)*** *$\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measumble, and*

(ii) ***(Partial averaging)***

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega)d\mathbb{P}(\omega) = \int_A X(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{G}. \tag{2.3.17}$$

If $\mathcal{G}$ is the σ-algebra generated by some other random variable W (i.e., $\mathcal{G} = \sigma(W)$), we generally write $\mathbb{E}[X|W]$ rather than $\mathbb{E}[X|\sigma(W)]$.

Using the introduced notation, we have $\forall A \in \sigma(X)$, $A = \{X \in B\}$,

$$\begin{aligned}
\int_A g(X)d\mathbb{P} &= \int_{\{\omega\in\Omega:\, X\in B\}} g(X(\omega))d\mathbb{P}(\omega) \\
&= \int_\Omega \mathbb{I}_{\{X\in B\}} g(X(\omega))d\mathbb{P}(\omega) \\
&= \int_{-\infty}^{\infty} \mathbb{I}(x \in B) g(x) f_X(x)dx \\
&= \int_{-\infty}^{\infty} \mathbb{I}(x \in B) \left[\int_{-\infty}^{\infty} y \frac{f_{X,Y}(x,y)}{f_X(x)}dy\right] f_X(x)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{I}(x \in B) y \frac{f_{X,Y}(x,y)}{f_X(x)} f_X(x)dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{I}(x \in B) y f_{X,Y}(x,y)dxdy \\
&\overset{(2.6.3)}{=} \mathbb{E}[\mathbb{I}_{\{X\in B\}} Y] \\
&\overset{(2.6.2)}{=} \mathbb{E}[\mathbb{I}_A Y] \\
&= \int_A Y d\mathbb{P},
\end{aligned}$$

which completes the proof.

Ex. 2.4 (Shreve)

In Example 2.2.8, X is a standard normal random variable and Z is an independent random variable satisfying

$$\mathbb{P}\{Z = 1\} = \mathbb{P}\{Z = -1\} = \frac{1}{2}.$$

We defined $Y = XZ$ and showed that Y is standard normal. We established that although X and Y are uncorrelated, they are not independent. In this exercise, we use moment-generating functions to show that Y is standard normal and X and Y are not independent[2]*.*

(i) Establish the joint moment-generating function formula

$$\mathbb{E}e^{uX+vY} = e^{\frac{1}{2}(u^2+v^2)} \cdot \frac{e^{uv} + e^{-uv}}{2}. \tag{2}$$

There are at least two ways to show this.

1° By the definition,

$$\begin{aligned}
\mathbb{E}e^{uX+vY} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ux+vxz} d\mu_Z(z) d\mu_X(x) \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2}e^{ux+vx\cdot 1} + \frac{1}{2}e^{ux+vx\cdot(-1)}\right] d\mu_X(x) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{(u+v)x} d\mu_X(x) + \frac{1}{2}\int_{-\infty}^{\infty} e^{(u-v)x} d\mu_X(x) \\
&= \mathbb{E}e^{(u+v)X} + \mathbb{E}e^{(u-v)X} \\
&= \frac{1}{2}e^{\frac{(u+v)^2}{2}} + \frac{1}{2}e^{\frac{(u-v)^2}{2}} \\
&= e^{\frac{u^2+v^2}{2}} \cdot \frac{e^{uv} + e^{-uv}}{2}.
\end{aligned}$$

[2] Note: it is a fact that independent random variables are uncorrelated. The converse is not true, *even for normal random variables*, although it is true of *jointly normal random variables* (cf. Shreve, p. 62, the comment to thm. 2.2.7 and example 2.2.10).

2° Using the law of total probability,

$$\begin{aligned}
\mathbb{E}e^{uX+vY} &= \mathbb{E}e^{uX+vXZ} \\
&= \mathbb{E}\left[e^{uX+vXZ}|Z=1\right]\mathbb{P}(Z=1) + \mathbb{E}\left[e^{uX+vXZ}|Z=-1\right]\mathbb{P}(Z=-1) \\
&= \frac{1}{2}\mathbb{E}e^{uX+vX} + \frac{1}{2}\mathbb{E}e^{uX-vX} \\
&= e^{\frac{u^2+v^2}{2}} \cdot \frac{e^{uv}+e^{-uv}}{2}.
\end{aligned}$$

(ii) Use the formula above to show that $\mathbb{E}e^{vY} = e^{\frac{1}{2}v^2}$. This is the moment generating function for a standard normal random variable, and thus Y must be a standard normal random variable.

Take $u = 0$ in (2). Then,

$$\mathbb{E}e^{vY} = e^{\frac{v^2}{2}} \cdot \frac{e^0+e^0}{2} = e^{\frac{v^2}{2}}.$$

Hence, Y is indeed a standard normal variable.

(iii) Use the formula in (i) and Theorem 2.2.7(iv) to show that X and Y are not independent.

Recall theorem concerning independent random variables.

Thm. 2.2.7. *Let X and Y be random variables. The following conditions are equivalent.*

(i) X and Y are independent.

(ii) The joint distribution measure factors:

$$\mu_{X,Y}(A \times B) = \mu_X(A) \cdot \mu_Y(B), \qquad \forall A, B \in \mathcal{B}(\mathbb{R}). \tag{2.2.8}$$

(iii) The joint cumulative distribution function factors:

$$F_{X,Y}(a,b) = F_X(a) \cdot F_Y(b), \qquad \forall a, b \in \mathbb{R}. \tag{2.2.9}$$

(iv) The joint moment-generating function factors:

$$\mathbb{E}e^{uX+vY} = \mathbb{E}e^{uX} \cdot \mathbb{E}e^{vY} \tag{2.2.10}$$

for all $u, v \in \mathbb{R}$ for which the expectations are finite.

(v) The joint density factors[3]:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y), \qquad \text{for almost every } x, y \in \mathbb{R}. \tag{2.2.11}$$

We have

$$\begin{aligned}
\mathbb{E}e^{uX} &= e^{\frac{u^2}{2}}, \\
\mathbb{E}e^{vY} &= e^{\frac{v^2}{2}}, \\
\mathbb{E}e^{uX} \cdot \mathbb{E}e^{vY} &= e^{\frac{u^2}{2}} \cdot e^{\frac{v^2}{2}} = e^{\frac{u^2+v^2}{2}},
\end{aligned}$$

hence

$$\mathbb{E}e^{uX+vY} = e^{\frac{u^2+v^2}{2}} \cdot \frac{e^{uv}+e^{-uv}}{2} \neq e^{\frac{u^2+v^2}{2}} = \mathbb{E}e^{uX} \cdot \mathbb{E}e^{vY}.$$

By theorem 2.2.7 (iv) X and Y cannot be independent.

[3] *If there is a joint density.*