# Measure theory and stochastic processes
# TA Session Problems No. 2

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Note: this is only a draft of the solutions discussed on Monday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

Note: the solution to Ex. 5 (additional) has been corrected!

## Ex. 1.8 (Shreve)

***(Moment-generating function).*** *Let $X$ be a nonnegative random variable, and assume that*

$$\varphi(t) = \mathbb{E}e^{tX}$$

*is finite for every $t \in \mathbb{R}$. Assume further that $\mathbb{E}\left[Xe^{tX}\right] < \infty$ for every $t \in \mathbb{R}$. The purpose of this exercise is to show that $\varphi'(t) = \mathbb{E}\left[Xe^{tX}\right]$ and, in particular, $\varphi'(0) = \mathbb{E}X$.*
*We recall the definition of derivative:*

$$\varphi'(t) = \lim_{s \to t} \frac{\varphi(t) - \varphi(s)}{t - s} = \lim_{s \to t} \frac{\mathbb{E}e^{tX} - \mathbb{E}e^{sX}}{t - s} = \lim_{s \to t} \mathbb{E}\left[\frac{e^{tX} - e^{sX}}{t - s}\right].$$

*The limit above is taken over a continuous variable $s$, but we can choose a sequence of numbers $\{s_n\}_{n=1}^{\infty}$ converging to $t$ and compute*

$$\lim_{s_n \to t} \mathbb{E}\left[\frac{e^{tX} - e^{s_n X}}{t - s_n}\right],$$

*where now we are taking a limit of the expectations of the sequence of random variables*

$$Y_n = \frac{e^{tX} - e^{s_n X}}{t - s_n}.$$

*If this limit turns out to be the same, regardless of how we choose the sequence $\{s_n\}_{n=1}^{\infty}$ that converges to $t$, then this limit is also the same as $\lim_{s \to t} \mathbb{E}\left[\frac{e^{tX} - e^{sX}}{t - s}\right]$ and is $\varphi'(t)$.*
*The Mean Value Theorem from calculus states that if $f(t)$ is a differentiable function, then for any two numbers $s$ and $t$, there is a number $\theta$ between $s$ and $t$ such that*

$$f(t) - f(s) = f'(\theta)(t - s).$$

*If we fix $\omega \in \Omega$ and define $f(t) = e^{tX(\omega)}$, then this becomes*

$$e^{tX(\omega)} - e^{sX(\omega)} = (t - s)X(\omega)e^{\theta(\omega)X(\omega)}, \tag{1.9.1}$$

*where $\theta(\omega)$ is a number depending on $\omega$ (i.e., a random variable lying between $t$ and $s$).*

*(i) Use the Dominated Convergence Theorem (Theorem 1.4.9) and equation (1.9.1) to show that*

$$\lim_{n \to \infty} \mathbb{E}Y_n = \mathbb{E}\left[\lim_{n \to \infty} Y_n\right] = \mathbb{E}\left[Xe^{tX}\right].$$

*This establishes the desired formula $\varphi'(t) = \mathbb{E}\left[Xe^{tX}\right]$.*

First, recall the **Dominated Convergence Theorem**.

**Thm. 1.4.9.** *Let $X_1, X_2, \ldots$ be a sequence of random variables converging almost surely to a random variable $X$. If there is another random variable $Y$ such that $\mathbb{E}Y < \infty$ and $|X_n| \leq Y$ almost surely for every $n$, then*

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X.$$

Now, from (1.9.1) and the definition of $Y_n$ we have

$$|Y_n| = \left|\frac{e^{tX} - e^{s_n X}}{t - s_n}\right| = \left|Xe^{\theta_n X}\right| \overset{(*)}{=} Xe^{\theta_n X} \leq Xe^{2|t|X}. \tag{1}$$

where $(*)$ is because of nonnegativity of $X$. The last inequality comes from the fact that $\theta_n \in [s_n, t]$ (or $\theta_n \in [t, s_n]$) and for $n$ large enough $\theta_n \leq \max\{t, s_n\} \leq 2|t|$ because we took $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to t$.
As we assumed that $\mathbb{E}\left[Xe^{tX}\right] < \infty$, $\forall t \in \mathbb{R}$, we can apply the Dominated Convergence Theorem to the sequence of $Y_n$, dominated by (integrable) $Xe^{2tX}$, to obtain

$$\varphi'(t) = \lim_{n\to\infty} \mathbb{E}Y_n = \mathbb{E}\left[\lim_{n\to\infty} Y_n\right] = \mathbb{E}\left[Xe^{tX}\right],$$

which completes the proof.

(ii) *Suppose the random variable $X$ can take both positive and negative values and $\mathbb{E}e^{tX} < \infty$ and $\mathbb{E}\left[|X|e^{tX}\right] < \infty$ for every $t \in \mathbb{R}$. Show that once again $\varphi'(t) = \mathbb{E}\left[Xe^{tX}\right]$. (Hint: Use the notation (1.3.1) to write $X = X^+ - X^-$.)*

Recall the notation 1.3.1. defining the **positive and negative parts of** $X$:

$$X^+(\omega) = \max\{X(\omega), 0\}, \quad X^-(\omega) = \max\{-X(\omega), 0\}. \tag{1.3.1}$$

If $X$ can take both positive and negative values, then, as indicated in the task, write $X = X^+ - X^-$, so $X$ becomes a difference of two *positive* random variables.
Then, similarly as in $(i)$, by the Mean Value Theorem, there exists $\theta_n \in [s_n, t]$ (or $\theta_n \in [t, s_s]$) such that

$$Y_n = Xe^{\theta_n X}$$

and for sufficiently large $n$

$$|Y_n| = \left|Xe^{\theta_n X}\right| \leq |X|e^{\theta_n X} \leq |X|e^{2|t||X|}.$$

So, to apply the Dominated Convergence Theorem, we need to show that $\mathbb{E}\left[|X|e^{t|X|}\right] < \infty$, $\forall t \in \mathbb{R}$. We have, $\forall t \in \mathbb{R}$,

$$\mathbb{E}\left[|X|e^{t|X|}\right] = \mathbb{E}\left[X^+e^{tX^+}\mathbb{I}_{\{X\geq 0\}}\right] + \mathbb{E}\left[X^-e^{tX^-}\mathbb{I}_{\{X<0\}}\right]. \tag{2}$$

Since we assumed $\mathbb{E}\left[|X|e^{tX}\right] < \infty$,$\forall t \in \mathbb{R}$, so

$$\mathbb{E}\left[|X|e^{tX}\right] = \mathbb{E}\left[X^+e^{tX^+}\mathbb{I}_{\{X\geq 0\}}\right] + \mathbb{E}\left[X^-e^{-tX^-}\mathbb{I}_{\{X<0\}}\right] < \infty,$$

which implies that $\forall t \in \mathbb{R}$

$$\mathbb{E}\left[X^+e^{tX^+}\mathbb{I}_{\{X\geq 0\}}\right] < \infty,$$
$$\mathbb{E}\left[X^-e^{-tX^-}\mathbb{I}_{\{X<0\}}\right] < \infty.$$

Or, equivalently[1], $\forall t \in \mathbb{R}$

$$\mathbb{E}\left[X^+e^{tX^+}\mathbb{I}_{\{X\geq 0\}}\right] < \infty,$$
$$\mathbb{E}\left[X^-e^{tX^-}\mathbb{I}_{\{X<0\}}\right] < \infty.$$

Thus, going back to (2) we can state that $\forall t \in \mathbb{R}$

$$\mathbb{E}\left[|X|e^{t|X|}\right] = \mathbb{E}\left[X^+e^{tX^+}\mathbb{I}_{\{X\geq 0\}}\right] + \mathbb{E}\left[X^-e^{tX^-}\mathbb{I}_{\{X<0\}}\right] < \infty.$$

Then also $\mathbb{E}\left[|X|e^{|t||X|}\right] < \infty$, $\forall t \in \mathbb{R}$.
Now, we can finally apply the Dominated Convergence Theorem to obtain that

$$\varphi'(t) = \lim_{n\to\infty} \mathbb{E}Y_n = \mathbb{E}\left[\lim_{n\to\infty} Y_n\right] = \mathbb{E}\left[Xe^{tX}\right].$$

[1]Indeed, because it has to hold for any $\forall t \in \mathbb{R}$, so for both positive and negative values of $t$.

## Ex. 1.15 (Shreve)

*Let $X$ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume $X$ has a density function $f(x)$ that is positive for every $x \in \mathbb{R}$. Let $g$ be a strictly increasing, differentiable function satisfying*

$$\lim_{y\to-\infty} g(y) = -\infty, \quad \lim_{y\to\infty} g(y) = \infty,$$

*and define the random variable $Y = g(X)$.*
*Let $h(y)$ be an arbitrary nonnegative function satisfying $\int_{-\infty}^{\infty} h(y)dy = 1$. We want to change the probability measure so that $h(y)$is the density function for the random variable $Y$. To do this, we define*

$$Z = \frac{h(g(X))g'(X)}{f(X)}. \tag{3}$$

*Now define $\tilde{\mathbb{P}}$ by*

$$\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

(i) *Show that $Z$ is nonnegative and $\mathbb{E}Z = 1$.*

We assumed that the density $f$ is positive $\forall x \in \mathbb{R}$, $h$ is nonnegative, and $g$ is strictly increasing, so its derivative $g'$ is positive. Hence, by its definition (3), clearly, $Z \geq 0$.

Next, we have

$$\begin{aligned}
\mathbb{E}Z &= \mathbb{E}\left[\frac{h(g(X))g'(X)}{f(X)}\right] \\
&= \int_{\mathbb{R}} \frac{h(g(X))g'(X)}{f(X)} f(x)dx \\
&= \int_{\mathbb{R}} h(g(X))g'(X)dx \\
&= \int_{\mathbb{R}} h(u)du \\
&= 1,
\end{aligned}$$

by the assumption that $h$ integrates to 1.

(ii) *Show that $Y$ has density $h$ under $\tilde{\mathbb{P}}$.*

We defined $\tilde{\mathbb{P}}$ as

$$\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

First, let us check what the cumulative distribution function of $Y$ under $\tilde{\mathbb{P}}$ looks like, since *if* $Y$ has density $h$ under $\tilde{\mathbb{P}}$

$$\begin{aligned}
\tilde{\mathbb{P}}(Y \leq a) &= \int_{\{g(X)\leq a\}} \frac{h(g(X))g'(X))}{f(X)} dP \\
&= \int_{-\infty}^{g^{-1}(a)} \frac{h(g(x))g'(x)}{f(x)} f(x)dx \\
&= \int_{-\infty}^{g^{-1}(a)} h(g(x))g'(x)dx \\
&= \int_{-\infty}^{g^{-1}(a)} h(g(x))dg(x) \\
&= \int_{-\infty}^{a} h(u)du,
\end{aligned}$$

where the last step comes from the change of variable formula, the last. Hence, under $\tilde{\mathbb{P}}$ the random variable $Y = g(X)$ has density $h$.

## Ex. 7 (additional)

*Let $X$ be a nonnegative random variable. Show that $\int X d\mathbb{P} \geq \frac{\mathbb{P}(X>1/n)}{n}$. Assume further that $\int X d\mathbb{P} = 0$. Show that it follows that $\mathbb{P}(X = 0) = 1$.*

**Lemma: (Markov's inequality)** *If $X$ is a nonnegative integrable random variable and $a > 0$, then*

$$\mathbb{E}X \geq a\mathbb{P}(\{X \geq a\}).$$

*Proof.* For any event $A$ consider the indicator random variable of this event, i.e.

$$\mathbb{I}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\mathbb{I}_{\{X>a\}} = 1$ if the event $\{X > a\}$ occurs and $\mathbb{I}_{\{X>a\}} = 0$ if $\{X \leq a\}$. So, for $a > 0$ we have

$$X \geq a\mathbb{I}_{\{X>a\}}, \tag{4}$$

since our indicator random variable can attain only two values: if the event $\{X > a\}$ occurs, then $\mathbb{I}_{\{X>a\}} = 1$ and so

$$X > a = a\mathbb{I}_{\{X>a\}};$$

if the event $\{X \leq a\}$ occurs, then $\mathbb{I}_{\{X>a\}} = 0$, hence

$$X > 0 = a\mathbb{I}_{\{X>a\}}.$$

Next, take the expectation on both sides of (4), which cannot reverse the inequality as the expectation is a monotonically increasing function. We have

$$\mathbb{E}X \geq \mathbb{E}\left[a\mathbb{I}_{\{X>a\}}\right]. \tag{5}$$

Next, due to linearity of expectation, we can rewrite the RHS in (5) as follows

$$\mathbb{E}\left[a\mathbb{I}_{\{X>a\}}\right] = a\mathbb{E}\left[\mathbb{I}_{\{X>a\}}\right] = a\left[1 \cdot \mathbb{P}(\{X > a\}) + 0 \cdot \mathbb{P}(\{X \leq a\})\right] = a\mathbb{P}(\{X > a\}).$$

Finally, we have

$$\mathbb{E}X \geq a\mathbb{P}(\{X > a\}),$$

which completes the proof. □

To show the fist part of the question notice that $\int X d\mathbb{P} = \mathbb{E}X$, so that it suffices to use Markov's inequality with $a = \frac{1}{n}$.

In the second part of the question we need to show that if the integral of a nonnegative function is equal to zero, then this function is zero almost everywhere. In probabilistic parlance, if a nonnegative random variable has zero expected value, then it has to be equal to zero almost surely.

To show this, consider the set $A = \{\omega \in \Omega : X(\omega) > 0\}$. It is equal to the union of an increasing sequence of sets $A_n = \{\omega \in \Omega : X(\omega) > 1/n\}$, $n = 1, 2, \dots$, i.e.

$$A = \{\omega \in \Omega : X(\omega) > 0\} = \bigcup_{n=1}^{\infty} \{\omega \in \Omega : X(\omega) > 1/n\} = \bigcup_{n=1}^{\infty} A_n.$$

Hence

$$0 \leq \frac{1}{n}\mathbb{P}(A_n) \leq \int_{A_n} X d\mathbb{P} \leq \int_A X d\mathbb{X} = 0,$$

from which follows that $\mathbb{P}(A_n) = 0$. Next, due to the monotone continuity of probability[2], we have

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = 0,$$

which completes the proof.

[2]Recall: if $(A_n)_{n=1}^{\infty}$ is an increasing sequence of sets, i.e. $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \dots$, such that $\bigcup_{n=1}^{\infty} A_n = A$, then $\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n)$ (similarly for a decreasing sequence of sets).

# Ex. 5 (additional)[3][4]

*Consider the setting of Theorem 1.6.1. Show that* $\tilde{\mathbb{P}}$ *and* $\mathbb{P}$ *are equivalent iff* $\mathbb{P}(Z > 0) = 1$.

First, recall theorem 1.6.1.

**Thm. 1.6.1.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $Z$ *be an almost surely nonnegative random variable with* $\mathbb{E}Z = 1$. *For* $A \in \mathcal{F}$, *define*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega). \tag{1.6.3}$$

*Then* $\tilde{\mathbb{P}}$ *is a probability measure. Furthermore, if* $X$ *is a nonnegative random variable, then*

$$\tilde{\mathbb{E}}X = \mathbb{E}[XZ].$$

*If* $Z$ *is almost surely strictly positive, we also have*

$$\mathbb{E}Y = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right] \tag{1.6.5}$$

*for every nonnegative random variable* $Y$.

Next, recall the definition of **equivalence of probabilistic measures**.

**Def. 1.6.3.** *Let* $\Omega \neq \emptyset$ *and* $\mathcal{F}$ *a* $\sigma$*-algebra of subsets* $\Omega$. *Two probability measures* $\mathbb{P}$ *and* $\tilde{\mathbb{P}}$ *on* $(\Omega, \mathcal{F})$ *are said to be equivalent, denoted* $\mathbb{P} \sim \tilde{\mathbb{P}}$, *if they agree which sets in* $\mathcal{F}$ *have probability zero.*

In other words,

$$\mathbb{P} \sim \tilde{\mathbb{P}} \iff \Big(\mathbb{P}(A) = 0 \Leftrightarrow \tilde{\mathbb{P}}(A) = 0, \;\; \forall A \in \mathcal{F}\Big).$$

*Proof.* (of the statement in the exercise)

$\Leftarrow$ Suppose that $\mathbb{P}(Z > 0) = 1$.

First, let $A \in \mathcal{F}$ be such that $\mathbb{P}(A) = 0$. Then the random variable $\mathbb{I}_A Z$ is almost surely zero under $\mathbb{P}$, hence, by the definition of $\tilde{\mathbb{P}}$ given by (1.6.3) we have

$$\tilde{\mathbb{P}}(A) = \int_\Omega \mathbb{I}_A(\omega) Z(\omega) d\mathbb{P}(\omega) = 0.$$

Next, let $B \in \mathcal{F}$ satisfy $\tilde{\mathbb{P}}(B) = 0$. Since we assume $\mathbb{P}(Z > 0) = 1$, so $\mathbb{P}(Z = 0) = 0$, and then, by (1.6.3) we have

$$\begin{aligned}\tilde{\mathbb{P}}(Z = 0) &= \int_{\{Z=0\}} Z d\mathbb{P} \\ &= \int_\Omega \mathbb{I}_{\{Z=0\}} Z d\mathbb{P} \\ &= 0.\end{aligned}$$

Hence, also under $\tilde{\mathbb{P}}$ the random variable $Z$ is almost surely positive and we can divide by $Z$ under $\tilde{\mathbb{P}}$. Then the random variable $\frac{1}{Z}\mathbb{I}_B = 0$ almost surely under $\tilde{\mathbb{P}}$ and by (1.6.5) we have

$$\mathbb{E}\mathbb{I}_B = \tilde{\mathbb{E}}\left[\frac{1}{Z}\mathbb{I}_B\right] = 0,$$

so

$$\mathbb{P}(B) = \mathbb{E}\mathbb{I}_B = 0.$$

Hence, $\mathbb{P}$ and $\tilde{\mathbb{P}}$ agree which sets have zero probability measure, i.e. $\mathbb{P} \sim \tilde{\mathbb{P}}$.

[3] Not discussed in the class.
[4] Corrected!

⇒ Suppose $\mathbb{P} \sim \tilde{\mathbb{P}}$, so that $\mathbb{P}(A) = 0 \Leftrightarrow \tilde{\mathbb{P}}(A) = 0, \ \forall A \in \mathcal{F}$.

Consider the following sets

$$\begin{aligned} B &= \{\omega \in \Omega : \ Z(\omega) = 0\}, \\ B^C &= \{\omega \in \Omega : \ Z(\omega) > 0\}, \end{aligned}$$

so that indeed $\Omega = B \cup B^C$, because we assumed that $\mathbb{P}(Z \geq 0) = 1$.

Then, by (1.6.3) we have

$$\begin{aligned} \tilde{\mathbb{P}}(B) &= \int_B Z(\omega) d\mathbb{P}(\omega) \\ &= \int_\Omega \mathbb{I}_B Z(\omega) d\mathbb{P}(\omega) \\ &= \int_\Omega \mathbb{I}_{\{\omega \in \Omega: \ Z(\omega) = 0\}} Z(\omega) d\mathbb{P}(\omega) \\ &= 0, \end{aligned}$$

and due to the equivalence of measures $\mathbb{P}(B) = 0$. Hence,

$$\mathbb{P}(Z > 0) = \mathbb{P}(B^C) = 1 - \mathbb{P}(B) = 1 - 0 = 1,$$

which completes the proof.

□

Just for completeness - **Radon-Nikodym's theorem**.

**Thm. 1.6.7.** *Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be equivalent probability measures defined on $(\Omega, \mathbb{F})$. Then there exists an almost surely positive random variable $Z$ such that $\mathbb{E}Z = 1$ and for every $A \in \mathcal{F}$*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega).$$