Measure theory and stochastic processes
TA Session Problems No. 1

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Note: this is only a draft of the solutions discussed on Friday and might contain some typos or more or less imprecise statements. If you find some, please let me know.

Ex. 1 (additional)

Let $\mathcal{A} = \{A_1, A_2, A_3\}$ be non-empty sets that form a partition of a set Ω. Write down all elements of $\sigma(\mathcal{A})$.

Let B_1, B_2 be two subsets of such that $B_1 \cap B_2$ and $(B1 \cup B_2)^C$ are non-empty. Write down all elements of $\sigma(\{B_1, B_2\})$.

Note: in this exercise it is worth drawing a picture of the sets in question.

Recall definition 1.1.1. of a **σ-algebra**.

Def. 1.1.1. *Let $\Omega \neq \emptyset$ and let $\mathcal{F}$ be a collection of subsets of Ω. Then $\mathcal{F}$ is called a σ-algebra (σ-field) if:*

(i) $\emptyset \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$,

(iii) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Moreover, for $C \subset \Omega$ we defined $\sigma(C)$ as a *minimal* σ-algebra containing C, called a **σ-algebra generated by** C, i.e. the intersection of all σ-algebras containing C.

A **partition** of a set Ω is a collection of its subsets $\{S_i\}_{i\in I}$, with I being some index set, such that $\bigcup_{i\in I} S_i = \Omega$ and $S_i \cap S_j = \emptyset$, $\forall i \neq j$. Hence, since $\mathcal{A}$ is a partition of Ω, we know that a complement of any set created using the sets from $\mathcal{A}$ is just a sum of the remaining set from $\mathcal{A}$, e.g. $A_1^C = A_2 \cup A_3$ or $(A_1 \cup A_2)^C = A_3$. Thus

$$\sigma(\mathcal{A}) = \{\emptyset, \Omega, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3\}.$$

Next, since $B_1 \cap B_2 \neq \emptyset$ and $(B_1 \cup B_2)^C \neq \emptyset$, we have

$$\sigma(\{B_1, B_2\}) = \left\{\emptyset, \Omega, B_1, B_1^C, B_2, B_2^C, B_1 \cap B_2, (B_1 \cap B_2,)^C, B_1 \cup B_2, (B_1 \cup B_2)^C\right\}.$$

Ex. 2 (additional)

Let Ω be a nonempty set and let for each i in some (index) set I $\mathcal{F}_i$ be a σ-algebra on Ω. Let $\mathcal{C}$ be some collection of subsets of Ω. In alternative wordings compared to Section A.2, but in content the same, we define $\sigma(\mathcal{C})$ to be the smallest σ-algebra that contains $\mathcal{C}$, i.e. the intersection of all σ-algebras that contain $\mathcal{C}$.

(a) *Show that $\bigcap_{i\in I} \mathcal{F}_i$ (the intersection of all σ-algebras $\mathcal{F}_i$) is a σ-algebra.*

Denote the intersection in question by $\mathcal{F}$, i.e.

$$\mathcal{F} := \bigcap_{i\in I} \mathcal{F}_i = \{A \subset \Omega :\ A \in \mathcal{F}_i,\ \forall i \in I\}.$$

We simply need to check that $\mathcal{F}$ satisfies three properties from the definition of a σ-algebra.

(i) $\emptyset \overset{?}{\in} \mathcal{F}$
Since each $\mathcal{F}_i$ is a σ-algebra, it has to contain the empty set, i.e. $\emptyset \in \mathcal{F}_i$, $\forall i \in I$. Thus, by construction, $\emptyset \in \mathcal{F}$, so *(i)* - checked;

(ii) $A \in \mathcal{F} \overset{?}{\Rightarrow} A^C \in \mathcal{F}$
Let $A \in \mathcal{F}$, which means $\forall i \in I$ we have $A \in \mathcal{F}_i$. Similarly as above, since each $\mathcal{F}_i$ is a σ-algebra, together with a set A, it has to contain its complement A^C. Thus, $A^C \in \mathcal{F}_i\ \ \forall i \in I$, which means that, indeed, $A^C \in \mathcal{F}$, so *(ii)* - checked.

(iii) $A_1, A_2, \dots \in \mathcal{F} \overset{?}{\Rightarrow} \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$
Let $A_1, A_2, \dots \in \mathcal{F}$. As previously, we have $A_1, A_2, \dots \in \mathcal{F}_i$, $\forall i \in I$, and, again, since all $\mathcal{F}_i$ are σ-algebras, they are closed under countable sums, meaning that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}_i$, $\forall i \in I$. Hence, $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$, which means that we have also checked *(iii)*.

(b) *Why is there is at least one σ-algebra that contains $\mathcal{C}$?*

Recall the two "extreme" examples of σ-algebras: $\{\emptyset, \Omega\}$ - the smallest σ-algebra and 2^{Ω} - the power set of Ω (the set of all subsets of Ω). Obviously, $\mathcal{C}$ needs to be contained in the latter σ-algebra, i.e. $\mathcal{C} \subset 2^{\Omega}$.

(c) *Here we take $\Omega = \mathbb{R}$. Argue that $\mathcal{B}(\mathbb{R})$ is equal to $\sigma(\mathcal{C})$, where $\mathcal{C} = \{(-\infty, a],\, a \in \mathbb{R}\}$.*

Recall that we have defined $\mathcal{B}(\mathbb{R})$, the **Borel σ-algebra on the real line**, as a σ-algebra generated by all open intervals on $\mathbb{R}$, call them $\mathcal{G} := \{(a, b) \subset \mathbb{R} :\ a < b,\, a, b \in \mathbb{R}\}$, i.e. $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{G})$. In other words, to get the Borel subsets of $\mathbb{R}$ start with the open intervals $(a, b) \subset \mathbb{R}$, and add all other sets that are necessary in order to have a σ-algebra[1].

Because we think of sets that can be *generated* (using the allowed σ-algebra operations) from all open intervals, to show the statement in question we need to show that the same sets can be "generated" out of the right-closed semi-lines using the same operations.

Because always $\mathcal{C} \subset \sigma(\mathcal{C})$ we have that all our right-closed semi-lines are in $\sigma(\mathcal{C})$. Take one such a semi-line $(-\infty, b]$ and we need its complement, (b, ∞) to also sit in $\sigma(\mathcal{C})$. Next, when we take an intersection of two semi-lines, say $(-\infty, b]$ and (a, ∞), with $a < b$, we have that an open-closed intervals $(a, b]$ need to be in $\sigma(\mathcal{C})$ as well.

Now, let us consider countable operations. We, have

$$
\begin{aligned}
(a, b) &= \bigcup_{i=1}^{\infty} \left(a, b - \frac{1}{n}\right], \\
[a, b] &= \bigcap_{i=1}^{\infty} \left(a - \frac{1}{n}, b\right], \\
[a, b) &= \bigcup_{i=1}^{\infty} \left[a, b - \frac{1}{n}\right],
\end{aligned}
$$

so we can generate out of intervals $(a, b]$ any intervals we wish using only countable unions and intersection. In particular, we can generate all open intervals, which are generators of the Borel σ-algebra on $\mathbb{R}$. Therefore, we can conclude that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$.

(d) *Consider a function $X : \Omega \to \mathbb{R}$. Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}$ that is such that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$. Suppose that all sets $\{X \in C\}$ (for $C \in \mathcal{C}$) belong to a σ-algebra $\mathcal{F}$ on Ω. Show that X is a random variable (Definition 1.1.5).*

Recall definition 1.1.5 of a **random variable**.

Def. 1.2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable is a real-valued function X defined on Ω with the property that for every Borel subset B of $\mathbb{R}$, the subset of Ω given by*

$$\{X \in B\} = \{\omega \in \Omega :\ X(\omega) \in B\}$$

is in the σ-algebra $\mathcal{F}$.

Hence, we need to show that a preimage of any Borel B under the function X, i.e. $\{X \in B\}$, is $\mathcal{F}$-measurable.

To start with, define

$$\mathcal{D} = \{B \subset \mathbb{R} :\ \{X \in B\} \in \mathcal{F}\},$$

the collection of all subsets of the real line which are $\mathcal{F}$-measurable. Notice that $\mathcal{D}$ is a σ-algebra. Indeed,

[1]Recall, σ-algebras are closed under taking of complements and countable sums, and, consequently, countable intersections.

(i) $\emptyset \overset{?}{\in} \mathcal{D}$
$\{X \in \emptyset\} = \emptyset$, which naturally is in $\mathcal{F}$ on Ω, as $\mathcal{F}$ is a σ-algebra and so needs to contain $\emptyset$; so *(i)* - checked;

(ii) $B \in \mathcal{D} \overset{?}{\Rightarrow} B^C \in \mathcal{D}$
let $B \in \mathcal{D}$, which means that $\{X \in B\} \in \mathcal{F}$; because $\mathcal{F}$ is a σ-algebra it has to contain $\{X \in B\}^C = \{X \notin B\} = \{X \in B^C\}$, so we end up with $B^C \in \mathcal{D}$, so *(ii)* - checked;

(iii) $B_1, B_2, \dots \in \mathcal{D} \overset{?}{\Rightarrow} \bigcup_{i=1}^{\infty} B_i \in \mathcal{D}$
let $B_1, B_2, \dots \in \mathcal{D}$ meaning that $\{X \in B_1\}, \{X \in B_2\}, \dots \in \mathcal{F}$; since $\mathcal{F}$ is a σ-algebra we know that $\bigcup_{i=1}^{\infty}\{X \in B_i\} \in \mathcal{F}$, so that $\bigcup_{i=1}^{\infty} B_i \in \mathcal{D}$ and *(iii)* is also checked.

Next, recall the lemma from the lecture:

Lemma: *Let $\mathcal{D}$ be σ-algebra on $\Omega \neq \emptyset$ and let $\mathcal{C}$ be some collection of subsets of Ω such that $\mathcal{C} \subset \mathcal{D}$. Then, $\sigma(\mathcal{C}) \subset \mathcal{D}$.*

Because we assumed that $\{X \in C\} \in \mathcal{F}$, $\forall C \in \mathcal{C}$, we know that $\mathcal{C} \subset \mathcal{D}$ and, by the lemma, $\sigma(\mathcal{C}) \subset \mathcal{D}$. Moreover, by assumption, $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, which means $\mathcal{B}(\mathbb{R}) \subset \mathcal{D}$. However, necessarily, $\mathcal{D} \subset \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$. Therefore, we arrive at

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{D} \subset \mathcal{B}(\mathbb{R}),$$

which means $\mathcal{D} = \mathcal{B}(\mathbb{R})$, i.e. the sets which are $\mathcal{F}$-measurable under the function X are basically all Borel sets. And this is the requirement for X to be a random variable, which completes the proof.

(e) *Suppose that for all $a \in \mathbb{R}$ the set $\{X \leq a\}$ is an element of $\mathcal{F}$. Show that X is a random variable.*

We need to show that if all the subsets of $\mathbb{R}$ of the form $(-\infty, a]$, i.e. right-closed semi-lines, are $\mathcal{F}$-measurable, then so do all the Borel sets $B \in \mathcal{B}(\mathbb{R})$. But this follows from the discussion above[2] and from the fact that $\mathcal{F}$-measurability means belonging to the σ-algebra $\mathcal{F}$ (which is closed under the required operations).

For instance, consider $\{X \leq a\} \in \mathcal{F}$. Then, also $\{X \leq a\}^C = \{X > a\} \in \mathcal{F}$. And so does

$$\{X \leq b\} \cap \{X > a\} = \{X \in (a, b]\} \in \mathcal{F},$$

for $a < b$. And we can proceed as in (c). Finally, notice that as a σ-algebra, $\mathcal{F}$ has to contain $\emptyset$, which is a preimage of the empty set, i.e. $\{X \in \emptyset\} = \emptyset \in \mathcal{F}$.

(f) *Suppose that for all $a \in \mathbb{R}$ the set $\{X < a\}$ is an element of $\mathcal{F}$. Is X a random variable?*

Yes, it is and the reasoning is similar to the one above, in (e).

Ex. 3 (additional)

Let μ_X be the distribution of a random variable X, see Definition 1.2.3. Show that μ_X is probability measure on the Borel sets of $\mathbb{R}$.

First, recall definition 1.1.2 of the **probability measure**.

Def. 1.1.2. *Let $\Omega \neq \emptyset$ and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0, 1]$, called the probability of A and written $\mathbb{P}(A)$ such that*

(i) $\mathbb{P}(\Omega) = 1$,

(ii) (countable additivity) whenever $A_1, A_2, \dots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Next, recall definition 1.2.3 of the **distribution measure**.

Def. 1.2.3. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution measure of X is the probability measure μ_X that assigns to each Borel subset B of $\mathbb{R}$ the mass $\mu_X(B) = \mathbb{P}(\{X \in B\})$.*

So we have to check that μ_X satisfies the two properties required for being a probability measure.

[2] Cf. (c) and $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$, with $\mathcal{C} = \{(-\infty, a),\, a \in \mathbb{R}\}$.

(i) $\mu_X(\mathbb{R}) = \mathbb{P}(\{X \in \mathbb{R}\}) = \mathbb{P}(\Omega) = 1$; *(i)*-checked.

(ii) (countable additivity) Let $B_1, B_2, \dots \in \mathcal{B}(\mathbb{R})$, $B_i \cap B_j = \emptyset$, $\forall i \neq j$. Then,

$$\mu_X\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathbb{P}\left(\left\{X \in \bigcup_{i=1}^{\infty} B_i\right\}\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} \{X \in B_i\}\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(\{X \in B_i\}\right) = \sum_{i=1}^{\infty} \mu_X(B_i),$$

so *(ii)*-checked.

Hence, we have shown that, indeed, the distribution measure μ_X is a probability measure.

Ex 1.5 (Shreve)

When dealing with double Lebesgue integrals, just as with double Riemann integrals, the order of integration can be reversed. The only assumption required is that the function being integrated be either nonnegative or integrable. Here is an application of this fact.
Let X be a nonnegative random variable with cumulative distribution function $F(x) = \mathbb{P}\{X \le x\}$. Show that

$$\mathbb{E}X = \int_0^{\infty} (1 - F(x))dx$$

by showing that

$$\int_{\Omega} \int_0^{\infty} \mathbb{1}_{[0,X(\omega))]}(x)dxd\mathbb{P}(\omega) \tag{1}$$

is equal to both $\mathbb{E}X$ and $\int_0^{\infty}(1 - F(x))dx$.

We assume that $X \ge 0$, so, according to what we are given in the exercise[3], we can reverse the order of integration in (1) and rewrite it as follows

$$\int_{\Omega} \int_0^{\infty} \mathbb{1}_{[0,X(\omega))]}(x)dxd\mathbb{P}(\omega) = \int_0^{\infty} \int_{\Omega} \mathbb{1}_{[0,X(\omega))]}(x)d\mathbb{P}(\omega)dx. \tag{2}$$

Then, the left hand-side (LHS) in (2) becomes

$$\int_{\Omega} \int_0^{\infty} \mathbb{1}_{[0,X(\omega))]}(x)dxd\mathbb{P}(\omega) = \int_{\Omega} \int_0^{X(\omega)} dxd\mathbb{P}(\omega) \overset{(*)}{=} \int_{\Omega} X(\omega)d\mathbb{P}(\omega) = \mathbb{E}X,$$

where the last step is due to the definition of the expected value of X. Notice that in $(*)$ we calculate the inner integral for a fixed ω and integrate over x's.

Next, the right hand-side (RHS) in (2) can be expressed as

$$\begin{aligned}
\int_0^{\infty} \int_{\Omega} \mathbb{1}_{[0,X(\omega))]}(x)d\mathbb{P}(\omega)dx &= \int_0^{\infty} \int_{\Omega} \mathbb{1}_{x \le X(\omega)} d\mathbb{P}(\omega)dx \\
&\overset{(**)}{=} \int_0^{\infty} \mathbb{P}(x \le X)dx \\
&= \int_0^{\infty} [1 - \mathbb{P}(X < x)]dx \\
&= \int_0^{\infty} [1 - F(x)]dx,
\end{aligned}$$

where the last step comes from the definition of the cumulative distribution function. Now, while tackling the inner integral in $(**)$ we have fixed x and integrated over ω's.

Hence, we have shown that

$$\mathbb{E}X = \text{LHS} = \text{RHS} = \int_0^{\infty} [1 - F(x)]dx,$$

which is the desired result.

[3] Which is basically Fubini's theorem.