Advanced Econometrics II
Homework Assignment No. 4

Deadline: 02.02.2015, 23:59

Submit to: `A.Borowska@tinbergen.nl`

Please submit your (typed) solution in a `pdf` file. Please motivate all your answers. For programming exercises (if there are any), the code has to be put, together with the main `pdf` solution file, in an archive file (e.g. `zip` or `rar`). Each code file shall contain your name, either as a comment or in its name.

Question 1

Consider the log-likelihood function

$$\ell(\theta|y) = \sum_{t=1}^{n} \ell(\theta|y^t),$$

and assume it is continuous in θ and has a unique maximum.

(a) *(Types of MLE)*
Discuss the difference between the Type 1 and Type 2 MLE. State the FOCs for the latter.

(b) *(Expectation of the score function)*
Prove that the score function has zero expectation, i.e.

$$\mathbb{E}_\theta g(\theta) = 0.$$

(c) *(Hessian and information matrices)*
Differentiate the identity

$$\int \exp(l_t(y^t,\theta)) \frac{\partial l_t(y^t,\theta)}{\partial \theta_i} dy_t = 0$$

with respect to θ_j to show that

$$\mathbb{E}_\theta(G_{ti}(y^t,\theta)G_{tj}(y^t,\theta) + (H)_{ij}(y^t,\theta)) = 0. \quad (1)$$

Hint: Apply the LIE to (1) conditioning on y_{t-1}.

(d) *(Information matrix equality)*
Prove the information matrix equality i.e.

$$\mathcal{I}(\theta) = -\mathcal{H}(\theta).$$

Hint: Use the result from (c).

(e) *(Asymptotic distribution)*
Derive the asymptotic distribution of the MLE. Use the fact the the score function asymptotically follows normal distribution with zero mean and variance given by the information matrix.
Hint: Apply the Taylor series expansion to the score function

Question 2

Consider the nonlinear regression model

$$y_t = \underbrace{\frac{\exp(\alpha t - \beta)}{1 + \exp(\alpha t - \beta)}}_{f_t(\alpha,\beta)} + \varepsilon_t, \qquad \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2),$$

for $t = 1, \dots, n$, with the data form the file `Data4.csv`.

Obtain the ML estimate for the parameters vector $\theta = (\alpha, \beta)^T$, together with the corresponding standard errors, using the **Newton-Raphson algorithm**, which in general is given by

1° Specify starting value θ_0 and set $k = 1$.

2° Repeat

$$\theta_k = \theta_{k-1} - \underbrace{\left(\frac{\partial^2 \ln g(y,\theta)}{\partial \theta} \bigg|_{\theta=\theta_{k-1}} \right)}_{H(\theta_{k-1})}^{-1} \frac{\partial \ln g(y,\theta)}{\partial \theta} \bigg|_{\theta=\theta_{k-1}},$$

$$k = k + 1,$$

until convergence (e.g. $||\theta_k - \theta_{k-1}|| < \epsilon$).

For convenience, you can apply the **Berndt-Hall-Hall-Hausman** (BHHH) method, where the Hessian matrix $H(\theta)$ is approximated with the outer product of gradients as follows

$$H(\theta) \approx \sum_{t=1}^{n} \left(\frac{\partial \ln g(y_t,\theta)}{\partial \theta} \right) \left(\frac{\partial \ln g(y_t,\theta)}{\partial \theta} \right)^T.$$

Hint: Here the whole procedure can be expressed for instance as follows:

1° Specify starting value θ_0 and set $k = 1$.

2° Repeat

$$\theta_k = \theta_{k-1} + \left(\dot{X}(\theta_{k-1})^T \dot{X}(\theta_{k-1}) \right)^{-1} \dot{X}(\theta_{k-1})^T \left(y - x(\theta_{k-1}) \right),$$

$$k = k + 1,$$

until $\max |\theta_k - \theta_{k-1}| < 0.0001$,

where $y = (y_1, \dots, y_n)^T$, $x(\theta) = (x_1(\theta), \dots, x_n(\theta))^T$ with $x_t(\theta) = f_t(\alpha, \beta)$ and $\dot{X}(\theta) = \left(\dot{X}_1(\theta)^T, \dots, \dot{X}_n(\theta)^T \right)^T$ with

$$\dot{X}_t(\theta) = \left(\frac{\partial f_t(\alpha,\beta)}{\partial \alpha}, \frac{\partial f_t(\alpha,\beta)}{\partial \beta} \right).$$